Knowledge Aggregators*

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Abstract

Given a group of agents with heterogeneous information, there are many potential forms for combining agents' information. This paper introduces some language for investigating different methods of aggregating information within groups, where the knowledge of each agent or group of agents takes the general operator form. The operator form allows for the modeling of agents who are less than perfectly rational in their information processing. Various specific aggregation formulations are considered in detail, including what 'somebody knows', what 'everybody knows', common knowledge, and distributed knowledge. Distributed knowledge is what can be determined by cooperating agents, given their information, under direct communication. Regularity and preservation properties of aggregators are investigated and applied to the specific aggregators listed.

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1 Introduction

The problem of combining attributes of multiple agents into a single attribute has a long history. The Condorcet Paradox, identified by Condorcet [7] in 1785, highlights the issues surrounding the aggregation of individuals' preferences. While each agent may have well-defined preferences, the group as a whole may end up with non-transitive preferences when pairwise preferences are elicited though majority voting. Demand aggregation, and the existence of a representative consumer, face similar issues. Samuelson argued in 1956 that *"Community indifference curves of the type needed for the derivation of community demand do not exist"* (Samuelson [27]). Gorman [14] gave explicit circumstances under which such demand aggregation is possible.

Similar problems arise in the context of incomplete information when attempting to aggregate individuals' knowledge structures. When knowledge partitions, as in Aumann [2], are taken as the primitives of the model, certain natural aggregations do not lead to group knowledge partitions. For example, suppose there are three possible outcomes to some process, say High, Medium, or Low. Ann knows if the outcome will be High, while Bob knows if it will be Low. The modeler can ask when does 'somebody know' the outcome. Clearly, somebody knows if the outcome will be High, or if it will be Low, but no-one knows if it will be Medium. The information structure necessary to convey what 'somebody knows' does not admit a representation in the usual partition framework of knowledge. This problem remains even if the set of available information structures is expanded to allow correspondence, or signal-based, knowledge. The correspondence, or signal, approach to information representation allows each state of the world to be associated with the set of states the agent considers possible at that state. In this example, when the state is Medium, Ann rules out High, while Bob rules out Low. Thus somebody rules out High and somebody rules out Low, but no-one rules out both High and Low. The information structure necessary to convey what 'somebody knows' does not even admit a representation in the more general correspondence framework of knowledge.

This problem can be addressed by using a more general notion as the primitive of the model; in particular using the knowledge operator approach. Throughout this work, each agent's knowledge is modeled by a knowledge operator, which links each event with the states at which the agent claims to know the event has occurred.

This paper introduces the concept of knowledge aggregators and proposes some methods of classifying these structures. The idea of an aggregator is to generate a single new knowledge operator based on the operators of a group of agents. A knowledge aggregator takes the operators from multiple agents and combines them into a new knowledge operator which describes the 'group' knowledge, for varying ideas of what 'group' knowledge might mean. Aggregators which are investigated in some detail include what 'somebody knows', what 'everybody knows', what is 'common knowledge', and what is 'distributed knowledge'.

Various properties of aggregators are proposed and investigated. Standard properties arising from viewing aggregators as functions of multiple inputs, such as anonymity, associativity, label neutrality, and identity, are considered. An aggregation is anonymous if changing the label on the agents does not change the aggregation, while the aggregator is label neutral if changing the label on the states does not change the aggregation. An aggregator is an identity if, when all agents have the same knowledge operator, the aggregation has this operator as well.

Properties related to relative informativeness are also developed. Aggregators are classified as positive or negative aggregators. Positive aggregators increase the amount of information known following the aggregation and can be associated with cooperative situations. Negative aggregators restrict the amount of information known following the aggregation, and are best associated with competitive situations. A simple criterion for identifying these properties is given.

Knowledge operators are often assumed in the literature to have certain regularity properties which can be interpreted as rationality assumptions, notably the various assumptions contained in Kripke's S5 system for Modal Logic.[1] Aggregators are said to preserve these rationality assumptions, or logic properties, if, whenever every agent has some aspect of rationality, then the aggregated information also has that aspect. Strong rationality assumptions, such as the partition assumption of Aumann [2] are considered, as well as a range of weaker assumptions. Generally speaking, the example aggregators considered tend to have fairly weak preservation properties.

An equivalent formulation for knowledge operators based on 'collections of known events' is introduced. In this formulation each state of the world is associated with the collection of events which an agent knows has occurred. Some results allow more direct proof when thought of in terms of these collections of known events, rather than using the knowledge operators directly.

These aggregation properties can also be viewed as helpful tools in the study of behavioral economics. Firstly, the weaker conditions imposed by knowledge operators, compared to partitions or correspondences, allow the existence of agents who are not rational in terms of information processing. The modeling presented allows the existence of agents who process information in a 'behavioral' manner. Secondly, because we can view an agent as being made of multiple competing selves, this modeling allows us to aggregate the information of each 'self' into a single entity. This justifies emergent behavior in which each 'self' can be fully

[1]Details on Model Logic can be found in Chellas [8]

rational, though the combined entity may not be.

Many other fields of study also investigate rationality and information aggregation, principally philosophy and computer science. The S5 logic system used in Kripke semantics is taken as an important case study in this work. In this paper, these ideas are able to be explored by appealing purely to set theory, allowing a wider audience in the economics field than most of the work in Modal Logic.

Information and knowledge concerns are very important in economics and game theory. From auctions to contract theory, the precise modeling of information is of paramount importance for the understanding of incentives and preferred strategies of strategic agents. Information concerns are critically relevant even in the absence of strategic considerations. Much has been written about decision making under uncertainty.[2]

Aumann [2] introduces the partitional approach to model knowledge and common knowledge. Geanakoplos [12] and [13], FLAG **combine references** and as Brandenburger et al [5] study the correspondence model. Samuelson [26] reviews how economists have modeled knowledge and information. Unawareness is treated in great detail in Dekel, Lipman, and Rustichini [9], Fagin and Halpern [10], and Heifetz, Meier, and Schipper [17].

Section 2 defines knowledge operators and aggregators, and introduces example aggregators to be investigated in subsequent sections. Section 3 considers properties of aggregators derived from aggregators being functions of multiple inputs, as well as those properties related to the relative informativeness of the agents' knowledge operators. Section 4 considers some rationality assumptions that can be made on the set of knowledge operators, and how these restrictions interact with knowledge aggregators. Section 5 gives an alternate formulation of knowledge operators for technical purposes. The appendix contains all proofs.

2 Knowledge Aggregation

A state space is a non-empty finite set Ω of outcome-relevant states. Let 2^Ω be the set of all subsets of Ω. There is a non-empty, finite list (ordered set) J of agents, with a typical

[2]Initial work in this field includes von Neumann and Morgenstern [30], Savage [28], and Anscombe and Aumann [1]. Machina [20] and [21] provide a survey of decision making under uncertainty.

agent being denoted i or j. Knowledge for each agent $j \in J$ is represented by a knowledge operator $K^j : 2^\Omega \to 2^\Omega$. The usual interpretation is that, for each event $E \in 2^\Omega$, the subset $K^j E$ contains the states at which agent j knows that event E has occurred. That is, agent j knows the event E has occurred at state ω precisely when $\omega \in K^j E$. Let $\mathcal{K}$ be the set of all possible knowledge operators, $\mathcal{K} = \{K : 2^\Omega \to 2^\Omega\}$. In a given model, different agents may have the same knowledge operators, while some knowledge operators may not be held by any agents.

Definition 1. *A tuple $(\Omega, J, \{K^j\}_{j \in J})$ is a knowledge model, where Ω is a state space, J is a finite list of agents, and each K^j is a function $K^j : 2^\Omega \to 2^\Omega$ which is the knowledge operator for agent $j \in J$.*

Under the current description, a knowledge operator is not restricted in any way. For example, it is entirely possible that an agent claims to know things which are not true. That is, $\omega \in K^j E$, but $\omega \notin E$. Similarly, it is possible that the agent knows the stock price tomorrow will be between \$100 and \$150, but not that the price will be between \$50 and \$200. No assumptions are made on the agent's information structures or processing ability at this time. Section 4 considers the effect of various rationality assumptions on knowledge aggregation.

2.1 General Knowledge Aggregation

A knowledge aggregator is a function that takes as input the knowledge operators of a number of agents, and produces a single new knowledge operator which represents the 'group knowledge' in some sense. The particular aggregator which may be of use to a modeler will depend on the problem at hand. In many game-theoretic situations, the aggregation of common knowledge is of great importance. If constructing a team project, where it is only important that somebody knows the relevant information, then a 'somebody knows' aggregation will be of greatest interest. Any function which takes in the information from multiple agents and constructs a new knowledge operator is a knowledge aggregator.

We are not concerned in this work with the method by which information is transferred between agents, algorithms which generate the group knowledge, or information revelation mechanisms. Our concern is exclusively with the final form and properties of group knowledge, or knowledge aggregation, which is obtained.

Definition 2. *Let Ω be a finite state space, and $\mathcal{K} = \{2^\Omega \to 2^\Omega\}$ be the set of functions from 2^Ω to 2^Ω. Let $\mathrm{Seq}(\mathcal{K})$ be the set of all finite sequences of elements of $\mathcal{K}$.*

A knowledge aggregator $\mathbb{A}$ is a function:

$$\mathbb{A} : \mathrm{Seq}(\mathcal{K}) \to \mathcal{K}$$

The sequence of knowledge operators represents the knowledge operators for some collection of agents. Let $J = (1, 2, \dots, |J|)$ be a finite list of agents and $(K^1, \dots, K^{|J|})$ be the sequence of knowledge operators associated with J. To simplify notation we write A^J to mean $\mathbb{A}(K^1, \dots, K^{|J|})$. It must be emphasized that where $\mathbb{A}$ is an aggregator, a function $\mathrm{Seq}(\mathcal{K}) \to \mathcal{K}$, the symbol A^J refers to a knowledge operator, a function $2^\Omega \to 2^\Omega$.

Definition 3. *Let $(\Omega, J, \{K^j\}_{j \in J})$ be a knowledge model, and $\mathbb{A} : \mathrm{Seq}(\mathcal{K}) \to \mathcal{K}$. The knowledge aggregation $A^J : 2^\Omega \to 2^\Omega$ is the knowledge operator given by*

$$A^J E = \mathbb{A}(K^1, \dots, K^{|J|})E, \text{ for all } E \in 2^\Omega$$

2.2 Classical Knowledge Aggregators

Four classical knowledge aggregators will be used throughout this work as example knowledge aggregators. These are the everybody knows aggregator, denoted $\curlywedge$, the common knowledge aggregator, denoted $\wedge$, somebody knows aggregator $\curlyvee$, and the distributed knowledge aggregator, denoted $\vee$. The intuitive idea of 'somebody knows' and 'everybody knows' are fairly self-explanatory. The common knowledge is well-studied in the economics literature, and is the concept that everybody knows, and everybody knows that everybody knows, and so on ad infinitum. Common Knowledge is discussed in much greater detail in Chapter 2.[3] Distributed knowledge is more familiar in the model logic literature, and describes what agents can know if they work together by directly and completely sharing information.

[3] FLAG chapter reference

We will usually be interested in how these knowledge aggregators behave on a given knowledge model $(\Omega, J, \{K^j\}_{j\in J})$. Often we will write the corresponding operators in the simplified forms: $\curlywedge^J$, $\wedge^J$, $\curlyvee^J$ and $\vee^J$.

Definition 4. *Fix a state space Ω, and let $\mathcal{K}$ be the associated set of knowledge operators.*

Define the everybody knows aggregator $\curlywedge : \mathrm{Seq}(\mathcal{K}) \to \mathcal{K}$ pointwise at each sequence $(K^1, \ldots, K^n) \in \mathrm{Seq}(\mathcal{K})$ and event $E \in 2^\Omega$, by

$$\curlywedge(K^1, \ldots, K^n)E = \bigcap_{i=1}^{n} K^i E$$

For a knowledge model $(\Omega, J, \{K^j\}_{j\in J})$, the everybody knows aggregation $\curlywedge^J : 2^\Omega \to 2^\Omega$ is

$$\curlywedge^J E = \bigcap_{j\in J} K^j E$$

Each state ω belongs to $\curlywedge^J E$ when, at ω, all agents $j \in J$ know event E.

In order to define common knowledge, let $(K)^{(s)}$ denote the operator K applied s times for any $s \in \mathbb{N}$.[4] Iterated application of this operator is possible as it has the same domain and codomain.

Definition 5. *Fix a state space Ω, and let $\mathcal{K}$ be the associated set of knowledge operators.*

Define the common knowledge aggregator $\wedge : \mathrm{Seq}(\mathcal{K}) \to \mathcal{K}$ pointwise at each sequence $(K^1, \ldots, K^n) \in \mathrm{Seq}(\mathcal{K})$ and event $E \in 2^\Omega$, by

$$\wedge(K^1, \ldots, K^n)E = \bigcap_{s=1}^{\infty} \left(\curlywedge(K^1, \ldots, K^n)\right)^{(s)} E$$

For a knowledge model $(\Omega, J, \{K^j\}_{j\in J})$, the common knowledge aggregation $\wedge^J : 2^\Omega \to 2^\Omega$ is

$$\wedge^J E = \bigcap_{s=1}^{\infty} \left(\curlywedge^J\right)^{(s)} E$$

Event E is common knowledge at ω if everybody knows E, and everybody knows that everybody knows E, and so on, ad infinitum.

[4] $\mathbb{N} = \{1, 2, 3, \ldots, \}$ throughout.

This is the definition of common knowledge as in, for example, Bacharach [3]. Alternative definitions for common knowledge are often used, as in, for example, Aumann [2]. Chapter 2 FLAG explores the distinction between these definitions. Both are generally equally useful definitions of common knowledge. For brevity, the work restricts attention to the Bacharach version of common knowledge presented in Definition 5.[5]

Definition 6. *Fix a state space Ω, and let $\mathcal{K}$ be the associated set of knowledge operators.*

Define the somebody knows aggregator $\Upsilon : \mathrm{Seq}(\mathcal{K}) \to \mathcal{K}$ pointwise at each sequence $(K^1, \ldots, K^n) \in \mathrm{Seq}(\mathcal{K})$ and event $E \in 2^\Omega$, by

$$\Upsilon(K^1, \ldots, K^n)E = \bigcup_{i=1}^{n} K^i E$$

For a knowledge model $(\Omega, J, \{K^j\}_{j \in J})$, the somebody knows aggregation $\Upsilon^J : 2^\Omega \to 2^\Omega$ is

$$\Upsilon^J E = \bigcup_{j \in J} K^j E$$

Each state ω belongs to $\Upsilon^J E$ when, at ω, at least one agent $j \in J$ knows event E.

Next we focus on distributed knowledge. Under distributed knowledge, the knowledge aggregation knows event E at state ω if, working collectively, the agents can determine that each state not in E has not occurred. It is an elimination process.

For an event $E \in 2^\Omega$, the complement of E is denoted $\neg E = \Omega \setminus E$. We say the agents can collectively determine that $\bar{\omega}$ has not occurred if there is an agent who knows some subset of $\neg\{\bar{\omega}\}$; that is, if there is an agent who knows an event which does not contain $\bar{\omega}$. The distributed knowledge of E is when all points outside of E are collectively known to have not occurred. This is formalized in Definition 7.

Definition 7. *Fix a state space Ω, and let $\mathcal{K}$ be the associated set of knowledge operators.*

Define the distributed knowledge aggregator $\vee : \mathrm{Seq}(\mathcal{K}) \to \mathcal{K}$ pointwise at each sequence $(K^1, \ldots, K^n) \in \mathrm{Seq}(\mathcal{K})$ and event $E \in 2^\Omega$, by

$$\vee (K^1, \ldots, K^n)E = \bigcap_{\bar{\omega} \notin E} \bigcup_{F \subset \neg\{\bar{\omega}\}} \bigcup_{i=1}^{n} K^i F \tag{1}$$

[5] The Bacharach definition of common knowledge was first introduced by Lewis [19].

For a knowledge model $(\Omega, J, \{K^j\}_{j\in J})$, the distributed knowledge aggregation $\vee^J : 2^\Omega \to 2^\Omega$ is

$$\vee^J E = \bigcap_{\bar{\omega}\notin E} \bigcup_{F\subset\neg\{\bar{\omega}\}} \bigcup_{j\in J} K^j F$$

Each state ω is in $\vee^J E$ when, for each state $\bar{\omega}$ outside E, there is some agent $j \in J$ and some event $F \not\ni \bar{\omega}$ such that j knows F. That is, for each state $\bar{\omega}$ outside E, there is some agent who knows that state $\bar{\omega}$ did not happen.

The distributed knowledge aggregator admits an alternative representation in terms of collections of known events, as detailed in Lemma 7 in Section 5. This alternative representation is useful for some of the proofs of properties of the distributed knowledge aggregator.

The next section considers some other properties of aggregators, and how they match with the aggregators defined above.

3 Core Properties of Knowledge Aggregators

This section proposes and analyses a selection of regularity properties of aggregators. Section 3.1 studies properties of knowledge aggregators arising due to their nature as functions which take a variable number of inputs. Section 3.2 considers properties related to the level of information held by the agents, and how this relates to the resulting aggregation for different aggregators.

3.1 Anonymity, Associativity, and Identity

This subsection studies properties of knowledge aggregators arising due to their nature as functions which take a variable number of inputs; primarily to do with the order of aggregation. Three properties are discussed: anonymity, associativity, and identity. Anonymity is the idea that the order in which agents are listed does not matter. Associativity is the idea that the order in which aggregation takes place does not matter. Identity is the idea that if all input operators are identical, then the aggregation may match each of the inputs.

While in general J has a defined order, in many cases it will be sufficient to consider J as a set of agents. In particular, when aggregator $\mathbb{A}$ is anonymous, as in Definition 8, then it is enough to think of J as a set. As usual, anonymity is the statement that the order of agents in the list does not matter.

Definition 8. *Fix a state space Ω, and let $\mathcal{K}$ be the associated set of knowledge operators.*

A knowledge aggregator $\mathbb{A} : \mathrm{Seq}(\mathcal{K}) \to \mathcal{K}$ is anonymous when, for all finite sequences of knowledge operators $(K^1, \ldots, K^n) \in \mathrm{Seq}(\mathcal{K})$ and all bijective functions $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$, then

$$\mathbb{A}(K^1, \ldots, K^n) = \mathbb{A}(K^{\sigma(1)}, \ldots, K^{\sigma(n)})$$

All of the classical knowledge aggregators in Section 2.2 are anonymous. Aggregators will fail anonymity when agents are treated differently. If, say, the modeler had a prior belief that Agent 1 was more trustworthy than the other agents, then an appropriate knowledge aggregation method would treat Agent 1 differently to the other agents.

While anonymity deals with the order of agents, the associativity condition is that the order of aggregation does not matter. We provide two equivalent definitions of associativity. The traditional definition of associativity for functions which take any finite sequence as input is given by Definition 9. Associativity, according to Definition 9, is the idea that aggregating the knowledge of all individuals in a single step is the same as aggregating any subset, then adding the other members to the aggregation afterwards. Intuitively speaking, it is a form of path-independence of aggregation.

Definition 10 provides an alternative definition of associativity. Associativity, according to Definition 10, states that given two or more subpopulations of the set of agents, taking the aggregation of all agents together should be the same as taking the aggregation of each subpopulation, and then aggregating the outcomes. Further, this process should not depend on the particular subpopulations used.

Proposition 1 shows that these definitions are equivalent.

Definition 9. *Fix a state space Ω, and let $\mathcal{K}$ be the associated set of knowledge operators.*

An aggregator $\mathbb{A} : \mathrm{Seq}(\mathcal{K}) \to \mathcal{K}$ is associative if, for any $(K^1, \ldots, K^n) \in \mathrm{Seq}(\mathcal{K})$, and cutoffs $1 \leq n_1 < n_2 \leq n$, then

$$\mathbb{A}(K^1, \ldots, K^n) = \mathbb{A}\left(K^1, \ldots, K^{n_1-1}, \mathbb{A}(K^{n_1}, \ldots, K^{n_2}), K^{n_2+1}, \ldots, K^n\right) \tag{2}$$

where $n_1 = 1$ is understood that $K^1, \ldots, K^{n_1-1}$ is the empty list, and similarly for $n_2 = n$.

Definition 10. *Fix a state space Ω, and let $\mathcal{K}$ be the associated set of knowledge operators.*

A knowledge aggregator $\mathbb{A} : \mathrm{Seq}(\mathcal{K}) \to \mathcal{K}$ is associative if for any sequence of knowledge operators $(K^1, \ldots, K^n) \in \mathrm{Seq}(\mathcal{K})$, and any collection of cutoffs $0 = n_0 < n_1 < n_2 < \cdots < n_k = n$, then aggregating directly over the full list $(K^1, \ldots, K^n)$, results in the same operator as is obtained by a two-step process, where first we aggregate all operators over those lists $K^{n_i+1}, \ldots, K^{n_{i+1}}$ separately, and then, in the second step, aggregate the k outcomes of the first step. Formally

$$\mathbb{A}(K^1, \ldots, K^n) = \mathbb{A}\left(\mathbb{A}(K^1, \ldots, K^{n_1}), \mathbb{A}(K^{n_1+1}, \ldots, K^{n_2}), \ldots, \mathbb{A}(K^{n_{k-1}+1}, \ldots, K^n)\right) \tag{3}$$

Proposition 1. *Fix a state space Ω, and let $\mathcal{K}$ be the associated set of knowledge operators.*

An aggregator $\mathbb{A} : \mathrm{Seq}(\mathcal{K}) \to \mathcal{K}$ satisfies Equation 2 if and only if it satisfies Equation 3. Therefore the definitions of associativity given in Definitions 9 and 10 are equivalent.

Associativity implies that the aggregator is an idempotent operator; that is, $A^J = A(A^J)$. The classical aggregators of everybody knows, somebody knows, and distributed knowledge are associative aggregators, as shown in Proposition 2. This is not the case for the common knowledge aggregator, as information is, in some sense, lost during the aggregation process. A counter-example showing common knowledge is not associative is given in Example 1.

Proposition 2. *Fix a state space Ω, and let $\mathcal{K}$ be the associated set of knowledge operators.*

The aggregators everybody knows, somebody knows, and the distributed knowledge are associative.

Example 1. *Let $J = \{1, 2\}$, $\Omega = \{a, b, c, d\}$. Suppose that:*

$$K^1\Omega = \{a, b, c\}, \quad K^1\{a, b, c\} = \{a\}, \quad K^1\{a\} = \{a\}, \quad K^1\{a, b\} = \{a, b\},$$

$$K^2\Omega = \{a, b\}, \quad K^2\{a, b, c\} = \{a, b\}, \quad K^2\{a, b\} = \{a, b\}, \quad K^2\{a\} = \{a\}.$$

Then:

$$\curlywedge^J\Omega = \{a, b, c\} \cap \{a, b\} = \{a, b\},$$

$$\curlywedge^J\{a,b\} = \{a,b\}.$$

Thus, for every $s \in \{1, 2, \cdots\}$*:*

$$\wedge^J\Omega = (\curlywedge^J)^{(s)}\Omega = \{a,b\}.$$

However, for $J_1 = \{1\}$*,* $J_2 = \{2\}$*, we find that* $\wedge^{J_1}\Omega = \{a\}$ *and* $\wedge^{J_2}\Omega = \{a,b\}$*. Therefore:*

$$\wedge(\wedge^{J_1}, \wedge^{J_2})\Omega = \{a\} \neq \{a,b\} = \wedge^J\Omega.$$

Therefore the common knowledge aggregator $\wedge : \mathrm{Seq}(\mathcal{K}) \to \mathcal{K}$ *is not associative.*

Where anonymity is concerned with the order of the agents, we can be equally concerned with the labeling of the states. A knowledge operator K will, in general, be sensitive to relabeling of the states, in the sense that for a given relabeling of the states f, then $K \circ f \neq f \circ K$. Aggregators, however, can certainly be independent of the labeling of the states. An aggregator is independent of the labeling of the states if relabeling the states, then applying the aggregator, is the same as applying the aggregator then relabeling the states. An aggregator which is independent of relabeling of the states is said to be label neutral.

Definition 11. *Fix a state space* Ω*, and let* $\mathcal{K}$ *be the associated set of knowledge operators.*

An aggregator $\mathbb{A} : \mathrm{Seq}(\mathcal{K}) \to \mathcal{K}$ *is label neutral if, for any bijection* $f : \Omega \to \Omega$*, and any* $(K^1, \ldots, K^n) \in \mathrm{Seq}(\mathcal{K})$*, then*

$$\mathbb{A} = f^{-1} \circ \left[\mathbb{A}\left(f \circ K^1, \ldots, f \circ K^n\right)\right]$$

where we make the usual abuse of notation that $f(E) = \{f(\omega) \mid \omega \in E\}$*.*

The classical aggregators somebody knows, everybody knows and distributed knowledge are label neutral. An aggregator will not be label neutral if it treats some states differently to others, or, potentially, if it includes some form of composition of knowledge operators. The aggregator in Example 3 is not label neutral as it treats states differently. The common knowledge aggregator is not label neutral as it includes composition of knowledge operators. An instance where the common knowledge aggregator is not label neutral is in Example 2.

Proposition 3. *Fix a state space Ω, and let $\mathcal{K}$ be the associated set of knowledge operators.*

The classical aggregators, everybody knows, somebody knows, and distributed knowledge are label neutral.

Example 2. *Let $J = \{1\}$, $\Omega = \{a, b, c\}$. Suppose that*

$$K^1\{a, b, c\} = \{a, b\},\ K^1\{a, b\} = \{a\},\ \text{and } K^1 E = E \text{ otherwise}$$

As $|J| = 1$, the everybody knows aggregation $\curlywedge^J$ is just $\curlywedge^J = K^1$. The common knowledge aggregation $\wedge^J$ has

$$\wedge^J \Omega = K^1\Omega \cap K^1K^1\Omega \cap \cdots = \{a, b\} \cap \{a\} \cap \cdots = \{a\}$$

Let $f : \Omega \to \Omega$ be a bijection, where $f(a) = b$, $f(b) = c$, and $f(c) = a$. The composition $f \circ K^1$ is somewhat ugly, but in particular,

$$(f \circ K^1)^{(4)}\Omega = \{b\},\ \text{and } (f \circ K^1)^{(5)}\Omega = \{c\}$$

Therefore, for $f(J) = (f \circ K^1) \in \mathrm{Seq}(\mathcal{K})$,

$$\begin{aligned} \wedge^{f(J)}\Omega &= \bigcap_{s=1}^{\infty} (f \circ K^1)^{(s)}\Omega \\ &\subset (f \circ K^1)^{(4)}\Omega \cap (f \circ K^1)^{(5)}\Omega \\ &= \{b\} \cap \{c\} = \emptyset \end{aligned}$$

As $\wedge^{f(J)}\Omega = \emptyset$, then $f^{-1} \wedge^{f(J)} \Omega \neq \wedge^J \Omega$. Thus the common knowledge aggregator $\wedge$ is not label neutral.

We would also like to know under what circumstances, if any, the aggregators have the property that when all agents have the same knowledge operator, then the aggregation returns just this same input aggregator. An aggregator is a k-identity if, for any collection of k agents with the same knowledge operator, then the aggregation is the same as the input operators. An aggregator is an identity if it is a k-identity for all k. More generally, for any subset of operators $\mathcal{L} \subset \mathcal{K}$, an aggregator is a k-identity on $\mathcal{L}$ if, for any collection of k

agents with the same knowledge operator, where that knowledge operator is in $\mathcal{L}$, then the aggregation is the same as the input operators. Similarly for an aggregator being an identity on $\mathcal{L}$.

Definition 12. *Fix a state space Ω, and let $\mathcal{K}$ be the associated set of knowledge operators.*

Let $\mathcal{L}$ be a collection of knowledge operators, $\mathcal{L} \subset \mathcal{K}$. A knowledge aggregator $\mathbb{A}$: $\mathrm{Seq}(\mathcal{K}) \to \mathcal{K}$ is a k-identity on $\mathcal{L}$ if

$$\mathbb{A}(\underbrace{L, L, \ldots, L}_{k \text{ times}}) = L$$

for all $L \in \mathcal{L}$. Aggregator $\mathbb{A}$ is an identity on $\mathcal{L}$ if it is a k-identity on $\mathcal{L}$ for all $k \in \mathbb{N}$.

Aggregator $\mathbb{A}$ is a k-identity if it is a k-identity on $\mathcal{K}$, and is an identity if it is a k-identity for all $k \in \mathbb{N}$.

This definition states that the restriction of the aggregator to $\mathrm{Seq}(\mathcal{L})$ is the identity. The simple classical aggregators of somebody knows and everybody knows are identities on the full set $\mathcal{K}$, while the more involved aggregators of common knowledge and distributed knowledge are not identities on all of $\mathcal{K}$. The sets $\mathcal{L}$ on which common knowledge and distributed knowledge are an identity are noted in Proposition 10.

3.2 Relative Informativeness

The four classical knowledge aggregators discussed admit a natural hierarchy of how much information they could be said to admit. Intuitively, it is very difficult for information to be common knowledge, and very easy for information to be available as distributed knowledge. It is much more likely that somebody knows a piece of information than that everybody knows it. This is given in Lemma 1.

Lemma 1. *Let $(\Omega, J, \{K^j\}_{j \in J})$ be a knowledge model. Then for every event $E \in 2^\Omega$, and every agent $j \in J$:*

$$\wedge^J E \subset \curlywedge^J E \subset K^j E \subset \curlyvee^J E \subset \vee^J E \tag{4}$$

This hierarchy suggests an idea of positive and negative aggregators. Positive aggregators are those that work to increase the amount that is known, while negative aggregators work to reduce the amount that is known.[6] The notion of relative informativeness of knowledge operators is given in Definition 13. Informativeness of knowledge operators is a partial order on the set of knowledge operators $\mathcal{K}$, as it is possible that agent 1 is well-informed in one area, while agent 2 is informed in another.

Definition 13. *Fix a state space Ω, and let $\mathcal{K}$ be the associated set of knowledge operators.*

Operator $K^1 \in \mathcal{K}$ is (weakly) more informative than operator $K^2 \in \mathcal{K}$ if $K^2E \subset K^1E$ for every event $E \in 2^\Omega$. Being "more informative than" is a partial order on the set of all knowledge operators.

Definition 14. *Fix a state space Ω, and let $\mathcal{K}$ be the associated set of knowledge operators.*

An aggregator $\mathbb{A} : \mathrm{Seq}(\mathcal{K}) \to \mathcal{K}$ is (weakly) positive if for all $(K^1, \ldots, K^n) \in \mathrm{Seq}(\mathcal{K})$, then $\mathbb{A}(K^1, \ldots, K^n)$ is more informative than K^i, for all $i = 1, \ldots, n$.

An aggregator $\mathbb{A} : \mathrm{Seq}(\mathcal{K}) \to \mathcal{K}$ is (weakly) negative if for all $(K^1, \ldots, K^n) \in \mathrm{Seq}(\mathcal{K})$, then K^i is more informative than $\mathbb{A}(K^1, \ldots, K^n)$, for all $i = 1, \ldots, n$.

From Lemma 1 the distributed knowledge, $\vee$, and somebody knows, $\curlyvee$, are positive aggregators; and common knowledge, $\wedge$, and everybody knows, $\curlywedge$, are negative aggregators. In fact, somebody knows, $\curlywedge$, is the least informative aggregator which is still positive, while everybody knows, $\curlyvee$, is the most informative aggregator which is still negative, as shown in Proposition 4.

Proposition 4. *Fix a state space Ω, and let $\mathcal{K}$ be the associated set of knowledge operators.*

An aggregator $\mathbb{A} : \mathrm{Seq}(\mathcal{K}) \to \mathcal{K}$ is positive if and only if for all $K_1, \ldots, K_n \in \mathrm{Seq}(\mathcal{K})$, then $\mathbb{A}(K_1, \ldots, K_n)$ is more informative than $\curlyvee(K_1, \ldots, K_n)$.

An aggregator $\mathbb{A} : \mathrm{Seq}(\mathcal{K}) \to \mathcal{K}$ is negative if and only if for all $K_1, \ldots, K_n \in \mathrm{Seq}(\mathcal{K})$, then $\curlywedge(K_1, \ldots, K_n)$ is more informative than $\mathbb{A}(K_1, \ldots, K_n)$.

[6] In applications in game theory, negative aggregators are most appropriate for modeling behavior in competitive games, while positive aggregators are more appropriate for cooperative games.

What happens to knowledge aggregations as new agents are added to the model? Positive and negative aggregators differ in what could be considered a null agent to add to the model; that is, an agent who makes no difference to the knowledge aggregation. In the case of a positive aggregator, an extra agent who knows nothing might be considered a null agent to add to the model. The knowledge operator associated with the agent who knows nothing is denoted $K^{\emptyset}$, and given by $K^{\emptyset}E = \emptyset$ for all events $E \in 2^{\Omega}$. Conversely, for negative aggregators, adding an agent who believes they know everything might be considered a null agent. The knowledge operator associated with the agent who believes that they know everything is denoted K^{Ω}, and given by $K^{\Omega}E = \Omega$ for all events $E \in 2^{\Omega}$. Positive and negative aggregators are called natural when their aggregation does not change due to the introduction of a null agent, as in Definition 15.

Definition 15. *Fix a state space Ω, and let $\mathcal{K}$ be the associated set of knowledge operators.*

A positive aggregator $\mathbb{A} : \mathrm{Seq}(\mathcal{K}) \to \mathcal{K}$ is naturally positive if $\mathbb{A}(K^1, \ldots, K^n, K^{\varnothing}) = \mathbb{A}(K^1, \ldots, K^n)$, for all $(K^1, \ldots, K^n) \in \mathrm{Seq}(\mathcal{K})$.

A negative aggregator $\mathbb{A} : \mathrm{Seq}(\mathcal{K}) \to \mathcal{K}$ is naturally negative if $\mathbb{A}(K^1, \ldots, K^n, K^{\Omega}) = \mathbb{A}(K^1, \ldots, K^n)$, for all $(K^1, \ldots, K^n) \in \mathrm{Seq}(\mathcal{K})$.

An aggregator $\mathbb{A} : \mathrm{Seq}(\mathcal{K}) \to \mathcal{K}$ is natural if it is either naturally negative, or naturally positive.

A cursory examination shows that the classical knowledge aggregators $\curlywedge$, $\wedge$, $\curlyvee$ and $\vee$ are all natural aggregators. An example of a synthetic aggregator which is positive but not natural is given in Example 3.

Example 3. *Let $\Omega = \{1, 2, 3, 4\}$, and let $\mathcal{K}$ be the associated set of knowledge operators. Define the knowledge aggregator $\mathbb{A}_* : \mathrm{Seq}(\mathcal{K}) \to \mathcal{K}$ so that for any sequence $(K^1, \ldots, K^n) \in \mathrm{Seq}(\mathcal{K})$, and event $E \in 2^{\Omega}$,*

$$\mathbb{A}_*(K^1, \ldots, K^n)E = (\{n\} \cap E) \cup \bigcup_{j=1}^{n} K^j E$$

That is, $\mathbb{A}_*$ asks whether any agent knows the event occurred, then adds in the number of agents, if that state is in the event. This is an entirely synthetic aggregator.

Clearly $\mathbb{A}_*(K^1, \ldots, K^n)E \supset \bigcup_{j=1}^n K^j E$. *By Proposition 4,* $\mathbb{A}_*$ *is positive. However,* $\mathbb{A}_*$ *is not natural in the sense of Definition 15 as, for example,*

$$\mathbb{A}_*(K^\emptyset)\Omega = \{1\}, \ \textit{while } \mathbb{A}_*(K^\emptyset, K^\emptyset)\Omega = \{2\}$$

Adding an additional agent with no information has changed the knowledge aggregation.

Aggregator $\mathbb{A}_*$ *is also clearly not label neutral as, for example, for any bijection* $f : \Omega \to \Omega$ *with* $f(1) \neq 1$, *then*

$$f^{-1} \circ \mathbb{A}_*(f \circ K^\emptyset)\Omega = f^{-1} \circ \mathbb{A}_*(K^\emptyset)\Omega = f^{-1}(\{1\}) \neq \{1\}$$

So $f^{-1} \circ \mathbb{A}_*(f \circ K^\emptyset) \neq \mathbb{A}_*(K^\emptyset)$, *and thus* $\mathbb{A}$ *is not label neutral.*

Knowledge aggregation can also be investigated when agents may learn new information. Fix a collection of agents J, and their knowledge operators $\{K^j : 2^\Omega \to 2^\Omega, \, j \in J\}$. Suppose that agents receive a positive knowledge shock, which weakly improves their knowledge. This positive shock transforms each operator K^j, $j \in J$, into a more informed operator $\tilde{K}^j$, $j \in J$.

Definition 16. *Fix a state space* Ω, *and let* $\mathcal{K}$ *be the associated set of knowledge operators.*

An aggregator $\mathbb{A} : \mathrm{Seq}(\mathcal{K}) \to \mathcal{K}$ *is increasing if for any pair of sequences* $(K^1, \ldots, K^n)$, $(\tilde{K}^1, \ldots, \tilde{K}^n) \in \mathrm{Seq}(\mathcal{K})$ *such that each operator* $\tilde{K}^i$ *is more informed than* K^i, *for each* $i \in \{1, \cdots, n\}$, *then the aggregation* $\mathbb{A}(\tilde{K}^1, \ldots, \tilde{K}^n)$ *is more informed than* $\mathbb{A}(K^1, \ldots, K^n)$. *Formally,*

$$\Big(K^i E \subset \tilde{K}^i E, \ \textit{for all } i \in \{1, \ldots n\}, \ E \subset \Omega\Big) \\ \implies \Big(\mathbb{A}(K^1, \ldots, K^n)E \subset \mathbb{A}(\tilde{K}^1, \ldots, \tilde{K}^n)E, \ \textit{for all } E \subset \Omega\Big).$$

While it might be expected that the four classical aggregators are all increasing, this is not the case. The everybody knows, somebody knows, and distributed knowledge aggregators are indeed all increasing, as in Lemma 2.

Lemma 2. *The aggregators everybody knows, somebody knows, and distributed knowledge are increasing operators.*

The common knowledge aggregator, however, is not increasing in general. Example 4 gives a pair of sequences of knowledge operators which demonstrates that the common knowledge aggregator is not increasing. The common knowledge aggregator is increasing if certain reasonable assumptions are made on the input knowledge operators. This is explored in Proposition 12.

Example 4. *Let* $\Omega = \{a, b, c\}$ *and operators* $K, \tilde{K} \in \mathcal{K}$ *given by*

$$K\{a, b, c\} = \{a\},\ K\{a, b\} = \emptyset,\ KE = E \text{ otherwise}$$
$$\tilde{K}\{a, b, c\} = \{a, b\},\ \tilde{K}\{a, b\} = \emptyset,\ \tilde{K}E = E \text{ otherwise}$$

Operators $K, \tilde{K}$ *differ only on* $\{a, b, c\}$ *where* $K\{a, b, c\} \subset \tilde{K}\{a, b, c\}$*. Thus, operator* $\tilde{K}$ *is more informed than operator* K*. However, as* $\tilde{K}\tilde{K}\{a, b, c\} = \emptyset$*, the corresponding common knowledge aggregations are*

$$\wedge(K)\{a, b, c\} = \{a\},\ \wedge(K)\{a, b\} = \emptyset,\ \wedge(K)E = E \text{ otherwise}$$
$$\wedge(\tilde{K})\{a, b, c\} = \emptyset,\ \wedge(\tilde{K})\{a, b\} = \emptyset,\ \wedge(\tilde{K})E = E \text{ otherwise}$$

Aggregations $\wedge(K), \wedge(\tilde{K})$ *differ only on* $\{a, b, c\}$ *where* $\wedge(K)\{a, b, c\} \supset \wedge(\tilde{K})\{a, b, c\}$*. Thus, operator* $\wedge(K)$ *is more informed than operator* $\wedge(\tilde{K})$*. Therefore the common knowledge aggregator is not an increasing aggregator in general.*

If an aggregator is natural and increasing, as is the case for many of our example aggregators, then the aggregator will behave nicely with the addition of new agents. In particular, for positive aggregators, new agents will mean aggregated information is more informative as more agents are added; while for negative aggregators, new agents will cause a reduction in aggregated information.

Proposition 5. *Fix a state space* Ω*, and let* $\mathcal{K}$ *be the associated set of knowledge operators. Let* $(K^1, \ldots, K^n) \in \mathrm{Seq}(\mathcal{K})$ *and* $m < n$*.*

If an aggregator $\mathbb{A} : \mathrm{Seq}(\mathcal{K}) \to \mathcal{K}$ *is naturally positive and increasing, then* $\mathbb{A}(K^1, \ldots, K^n)$ *is more informative than* $\mathbb{A}(K^1, \ldots, K^m)$*. Similarly, if* $\mathbb{A} : \mathrm{Seq}(\mathcal{K}) \to \mathcal{K}$ *is naturally negative and increasing, then* $\mathbb{A}(K^1, \ldots, K^m)$ *is more informative than* $\mathbb{A}(K^1, \ldots, K^n)$*.*

Finally we consider a notion of stationarity for knowledge aggregators, as a counterpart to the notion of naturalness given in Definition 15. An aggregator is natural if adding an extreme agent does not change the knowledge aggregation. Similarly, we say an aggregator is stationary if the aggregation does not change when adding a new agent who is both weakly less informed than some existing agent, and weakly more informed than an existing agent.

Definition 17. *Fix a state space Ω, and let $\mathcal{K}$ be the associated set of knowledge operators.*

An aggregator $\mathbb{A} : \mathrm{Seq}(\mathcal{K}) \to \mathcal{K}$ is stationary if for any sequence $J = (K^1, \ldots, K^n) \in \mathrm{Seq}(\mathcal{K})$, and any K^* such that K^* is less informed than K^{j_1} for some $j_1 \in J$, and is K^* is more informed than K^{j_2} for some $j_2 \in J$, then

$$\mathbb{A}(K^1, \ldots, K^n) = \mathbb{A}(K^1, \ldots, K^n, K^*)$$

Stationarity is closely related to the property of being an identity. In particular, if aggregator $\mathbb{A}$ is a 1-identity on some $\mathcal{L} \subset \mathcal{K}$, and is stationary, then it is an identity on $\mathcal{L}$.

Proposition 6. *Fix a state space Ω, and let $\mathcal{K}$ be the associated set of knowledge operators.*

Let $\mathcal{L} \subset \mathcal{K}$, and aggregator $\mathbb{A} : \mathrm{Seq}(\mathcal{K}) \to \mathcal{K}$ be a 1-identity on $\mathcal{L}$, and stationary. Then $\mathbb{A}$ is an identity on $\mathcal{L}$.

Each of the classical knowledge aggregators is stationary. However, fairly natural aggregators need not be stationary. For example, the 'at least two agents know' aggregator is not stationary.

Example 5. *Let $\Omega = \{1, 2, 3, 4\}$, and let $\mathcal{K}$ be the associated set of knowledge operators. Define the 'at least two agents know' knowledge aggregator $\mathbb{A}_2 : \mathrm{Seq}(\mathcal{K}) \to \mathcal{K}$ such that for any sequence $(K^1, \ldots, K^n) \in \mathrm{Seq}(\mathcal{K})$, then for each event $E \in 2^\Omega$,*

$$\mathbb{A}_2(K^1, \ldots, K^n)E = \bigcup_{\substack{i,j \in \{1,\ldots,n\} \\ i \neq j}} \left(K^i E \cap K^j E\right)$$

That is, $\mathbb{A}_2$ asks whether any pair of agents knows the event occurred.

Let $K \in \mathcal{K}$ be a knowledge aggregator with $K \neq K^\emptyset$. By construction $\mathbb{A}_2(K) = K^\emptyset$, while $\mathbb{A}_2(K, K) = K$. As K is both more informed and less informed than K, and $\mathbb{A}_2(K) \neq A_2(K, K)$, the aggregator $\mathbb{A}_2$ is not stationary.

In fact, the aggregator $\mathbb{A}_2$ is a k-identity for all $k \geq 2$; just not a 1-identity, and so not an identity aggregator.

4 Logic Properties of Knowledge Aggregators

There are a variety of assumptions often made to restrict the available knowledge operators to those which satisfy certain logical properties. The strongest assumption, which is also the most common, is to restrict knowledge operators to those which admit representation by a partition of the state space. In such a model, the state space is divided into partition, where the agent is unable to distinguish states in the same partition element, but can distinguish between partition elements. A weaker, but still common, assumption is the knowledge correspondence model. In this model of knowledge, in each state of the world the agent receives a signal which informs them of the the states they consider possible.

These restrictions on the state space are further broken down into the Modal Logic S5 system. This is a list of five rationality assumptions which together imply a partition representation, but which can also be investigated individually.

4.1 Partitions and Correspondences

Given a state space Ω, any partition π of the state space is a knowledge partition. A knowledge partition π is interpreted as, if $\omega^* \in \Omega$ is the true state of the world, then the agent believes that all states $\omega \in \pi(\omega^*)$ are possible, and that all states $\bar{\omega} \notin \pi(\omega^*)$ are not possible. Each knowledge partition π is associated with a knowledge operator K_π according to the formula

$$K_\pi E = \{\omega \in \Omega \mid \pi(\omega) \subset E\} \tag{5}$$

This equation says that if partition π represents the agent's information, then $K_\pi E$ should be the set of states where every state the agent considers possible is contained in the event E. We say an operator is partitional if there exists some partition which generates the operator using Equation 5.

Definition 18. *Fix a state space Ω, and let $\mathcal{K}$ be the associated set of knowledge operators.*

An operator $K \in \mathcal{K}$ is partitional if there exists some partition π of Ω such that $K = K_\pi$, where K_π is given by Equation 5. The set of partitional operators is denoted $\mathcal{K}_P$.

Remark 1. *A partitional knowledge operator will necessarily satisfy certain regularity conditions. In particular, any partitional knowledge operator $K : 2^\Omega \to 2^\Omega$ will satisfy $K\Omega = \Omega$, $K(E \cap F) = KE \cap KF$, $KE \subset E$, $KE \subset KKE$ and $\neg KE \subset K(\neg KE)$. Moreover, any knowledge operator satisfying these properties is partitional. This will be discussed further in Section 4.2.*

Turning to the question of knowledge aggregators; if the knowledge operators of all agents are partitional, which knowledge aggregations will also be partitional? This question has a surprising answer. The relatively simple aggregators of somebody knows, $\curlyvee$, and everybody knows, $\curlywedge$, do not preserve this property of being partitional. That is, even if all agents have the very strong rationality property of a partitional information structure, nonetheless very simple statements like 'does somebody know this' do not have such rationality properties. Example 6 provides a simple model where this is the case. Proposition 7 shows that the more developed aggregators of common knowledge, $\wedge$, and distributed knowledge, $\vee$, are partitional-preserving.

Definition 19. *Fix a state space Ω, and let $\mathcal{K}$ be the associated set of knowledge operators.*

An aggregator $\mathbb{A} : \mathrm{Seq}(\mathcal{K}) \to \mathcal{K}$ is partition-preserving if, for all sequences of partitional knowledge operators $(K^1, \ldots, K^n) \in \mathrm{Seq}(\mathcal{K}_P)$, the knowledge aggregation $\mathbb{A}(K^1, \ldots, K^n)$ is also a partitional operator.

Example 6. *Let $J = \{1, 2\}$, $\Omega = \{a, b, c\}$, and*

$$K^1\{a\} = K^1\{b\} = \emptyset, \quad K^1\{a, c\} = K^1\{b, c\} = \{c\}, \quad K^1E = E \text{ otherwise}$$
$$K^2\{b\} = K^2\{c\} = \emptyset, \quad K^2\{a, b\} = K^2\{a, c\} = \{a\}, \quad K^2E = E \text{ otherwise}$$

Operator K^1 is a partitional operator given by the partition $\pi^1 = \{\{a, b\}, \{c\}\}$ according to Equation 5. Similarly, operator K^2 is a partitional operator given by the partition $\pi^2 = \{\{a\}, \{b, c\}\}$.

The somebody knows aggregation $\curlyvee^J$ *has*

$$\curlyvee^J(\{a,b\} \cap \{b,c\}) = \curlyvee^J\{b\} = \emptyset, \textit{ while}$$
$$\curlyvee^J\{a,b\} \cap \curlyvee^J\{b,c\} = \{a,b\} \cap \{b,c\} = \{b\}$$

As $\curlyvee^J(\{a,b\} \cap \{b,c\}) \neq \curlyvee^J\{a,b\} \cap \curlyvee^J\{b,c\}$*, by Remark 1, the somebody knows aggregation* $\curlyvee^J$ *is not partitional in this case, and so the somebody knows aggregator* $\curlyvee$ *is not partitional in general. Similarly, the everybody knows aggregation* $\curlywedge^J$ *has*

$$\curlywedge^J\{a,b\} = \{a,b\} \cap \{a\} = \{a\}, \textit{ and}$$
$$\curlywedge^J\{a\} = \emptyset \cap \{a\} = \emptyset.$$

As $\curlywedge\{a,b\} \not\subset \curlywedge\curlywedge\{a,b\}$*, by Remark 1, the everybody knows aggregation* $\curlywedge^J$ *is not partitional in this instance. Thus, the everybody knows aggregator* $\curlywedge$ *is not partitional.*

Proposition 7. *The aggregators common knowledge and distributed knowledge are both partition-preserving aggregators.*

The correspondence knowledge framework is a middle ground between the more general universe of knowledge operators and the more restrictive universe of partitions. A knowledge correspondence is a function $\gamma : \Omega \to 2^{\Omega}$. The usual interpretation is that, at every state $\omega \in \Omega$, the agent considers as possible exactly those states in $\gamma(\omega)$, ruling out all states that do not belong to $\gamma(\omega)$. In other words, at state $\omega \in \Omega$, the agent thinks that an event E is sure to happen if and only if $\gamma(\omega) \subset E$, and is sure that event E does not happen if and only if $E \cap \gamma(\omega) = \emptyset$.

Each knowledge correspondence γ is associated with a knowledge operator K_γ according to the formula

$$K_\gamma E = \{\omega \in \Omega \mid \gamma(\omega) \subset E\} \tag{6}$$

This equation says that if correspondence γ represents the agent's information, then $K_\gamma E$ should be the set of states where every state the agent considers possible is contained in the event E. Definition 20 states that an operator is a correspondence operator if there exists some correspondence which generates the operator using Equation 6.

Definition 20. *Fix a state space Ω, and let $\mathcal{K}$ be the associated set of knowledge operators.*

An operator $K \in \mathcal{K}$ is a correspondence operator if there exists some correspondence $\gamma : \Omega \to 2^\Omega$ such that $K = K_\gamma$, where K_γ is given by Equation 6. The set of correspondence operators is denoted $\mathcal{K}_C$.

An operator $K : 2^\Omega \to 2^\Omega$ is a correspondence operator so long as it satisfies $K\Omega = \Omega$, and $K(E \cap F) = KE \cap KF$ for all events $E, F \in 2^\Omega$, as shown in Lemma 3. These properties are explored further in Section 4.2.

Lemma 3. *Fix a state space Ω, and let $\mathcal{K}$ be the associated set of knowledge operators.*

An operator $K \in \mathcal{K}$ is a correspondence operator if and only if $K\Omega = \Omega$ and $K(E \cap F) = KE \cap KF$ for all events $E, F \in 2^\Omega$.

The correspondence framework is a strict generalization of the partition framework. Lemma 4 shows that any knowledge partition is just a knowledge correspondence γ that satisfies the additional properties that:

P.1: $\omega \in \gamma(\omega)$, for every $\omega \in \Omega$.

P.2: For every $\omega, \omega' \in \Omega$, the sets $\gamma(\omega)$ and $\gamma(\omega')$ either coincide or have no element in common.

Lemma 4. *Fix a state space Ω, and let $\mathcal{K}$ be the associated set of knowledge operators.*

Let $\gamma : \Omega \to 2^\Omega$ be a knowledge correspondence, and suppose γ satisfies properties P.1 and P.2. Then, K_γ is partitional. In general, $\mathcal{K}_C \subset \mathcal{K}_P$.

Turning again to the question of knowledge aggregation; if the knowledge operators of all agents are correspondence operators, which knowledge aggregations will also be correspondence operators? As seen in Example 6, the somebody knows aggregator $\curlyvee$ does not preserve the property of being a correspondence operator, as in that example all input operators were partitional, so were also correspondence operators by Lemma 4, while the output operator failed to have $K(E \cap F) = KE \cap KF$, so was not a correspondence operator by Lemma 3. The remaining classical aggregators common knowledge, distributed knowledge and everybody knows all preserve the property of being a correspondence operator.

Definition 21. *Fix a state space Ω, and let $\mathcal{K}$ be the associated set of knowledge operators.*

An aggregator $\mathbb{A} : \mathrm{Seq}(\mathcal{K}) \to \mathcal{K}$ is correspondence-preserving if, for all sequences of correspondence knowledge operators $(K^1, \ldots, K^n) \in \mathrm{Seq}(\mathcal{K}_C)$, the knowledge aggregation $\mathbb{A}(K^1, \ldots, K^n)$ is also a correspondence operator.

Proposition 8. *The aggregators common knowledge, distributed knowledge, and everybody knows, are all correspondence-preserving aggregators.*

4.2 Modal Logic's S5 Axioms

The partitional and correspondence assumptions on knowledge operators are representation assumptions which do not, within their definition, contain clear information on what aspects of rationality are being assumed, nor do they make clear why we might think an agent would be partitional or have a correspondence operator. The S5 logic system is a series of rationality assumptions which are direct restrictions of the knowledge operator itself and are individually justifiable to some extent. While a modeler might hope that their agent will satisfy each of the S5 axioms, evidence from behavioral economics seems to suggest that these axioms are regularly broken in practice.

K.0 (Awareness). Let $\mathcal{K}_0 = \{K \in \mathcal{K} \mid K\Omega = \Omega\}$.

Awareness requires that, at every state, agent j knows that event Ω occurs. This property rules out the existence of subjective states. Agents know perfectly well the state space.

K.1 (Monotonicity). Let $\mathcal{K}_1 = \{K \in \mathcal{K} \mid K(E \cap F) \subset KE \cap KF,\ \ E, F \in 2^\Omega\}$.

Monotonicity is equivalent to the requirement that $E \subset F \Rightarrow K^jE \subset K^jF$, for all $E, F \in 2^\Omega$. This property states that if a agent knows a specific event has occurred, then she knows that a more general event has also occurred. Monotonicity will be one of the main assumptions questioned here.

K.2 (Conjunction). Let $\mathcal{K}_2 = \{K \in \mathcal{K} \mid K(E \cap F) \supset KE \cap KF,\ \ E, F \in 2^\Omega\}$.

Conjunction is a rationality condition on the agents. It states that if a agent knows that events E and F happen individually, then she knows that the specific event that is the intersection of E and F has indeed occurred.

K.3 (Truth). Consider $\mathcal{K}_3 = \{K \in \mathcal{K} \mid KE \subset E, \;\; E \in 2^\Omega\}$.

This property says that if agent j knows event E, then E occurs; that is, j can only learn truths.

K.4 (Transparency). Let $\mathcal{K}_4 = \{K \in \mathcal{K} \mid KE \subset KKE, \;\; E \in 2^\Omega\}$.

Some authors refer to this property as positive introspection. It indicates that agents are aware of their own knowledge.

K.5 (Wisdom). Let $\mathcal{K}_5 = \{K \in \mathcal{K} \,|\, \neg KE \subset K(\neg KE), \;\; E \in 2^\Omega\}$.

Some authors refer to this as negative introspection. It indicates that agents are aware of their own limitations.

If an agent's knowledge operator satisfies all the S5 axioms, then it is a partitional operator in the style of Definition 18. Similarly, any partitional operator will satisfy all the S5 axioms. This was mentioned earlier in Remark 1, and is proven as the main result of Bacharach [3].

Proposition 9. *(Bacharach's Partition Theorem [3]) A knowledge operator K satisfies all S5 axioms if and only if it is a partitional operator. That is*

$$\mathcal{K}_P = \mathcal{K}_0 \cap \mathcal{K}_1 \cap \mathcal{K}_2 \cap \mathcal{K}_3 \cap \mathcal{K}_4 \cap \mathcal{K}_5$$

Similarly, Lemma 3 states that an operator is a correspondence operator if and only if it satisfies Omniscience, Monotonicity, and Conjunction. That is

$$\mathcal{K}_C = \mathcal{K}_0 \cap \mathcal{K}_1 \cap \mathcal{K}_2$$

We promised earlier to give the collections $\mathcal{L} \subset \mathcal{K}$ on which the classical aggregators are an identity. As noted, the simple aggregators everybody knows and somebody knows are identities on all of $\mathcal{K}$. The common knowledge aggregator is an identity on $\mathcal{K}_4$. The distributed knowledge aggregator is an identity on $\mathcal{K}_C$.

Proposition 10. *Fix a state space Ω, and let $\mathcal{K}$ be the associated set of knowledge operators.*

Aggregators everybody knows and somebody knows are identities on $\mathcal{K}$. The common knowledge aggregator is an identity on $\mathcal{K}_4$. The distributed knowledge aggregator is an identity on $\mathcal{K}_C$.

In Section 4.1 we discussed whether aggregators had preservation properties in terms of partitional and correspondence operators. The same questions can be asked with regard to each of the S5 axioms. That is, if the knowledge operators of all agents satisfy any of the S5 properties, which knowledge aggregations will also satisfy this property? A related property is that of 'forcing'. A aggregator 'forces' a property if the aggregation necessarily has some property, even if the input operators do not have this property. More generally, we can talk about $\mathcal{L}$-preserving, and $\mathcal{L}$-forcing aggregators for any subset of knowledge operators $\mathcal{L} \subset \mathcal{K}$.

Definition 22. *Fix a state space Ω, let $\mathcal{K}$ be the associated set of knowledge operators, and $\mathcal{L} \subset \mathcal{K}$.*

An aggregator $\mathbb{A} : \mathrm{Seq}(\mathcal{K}) \to \mathcal{K}$ is $\mathcal{L}$-preserving if, for all sequences of knowledge operators $(K^1, \ldots, K^n) \in \mathrm{Seq}(\mathcal{L})$, the knowledge aggregation $\mathbb{A}(K^1, \ldots, K^n)$ is also in $\mathcal{L}$.

Definition 23. *Fix a state space Ω, let $\mathcal{K}$ be the associated set of knowledge operators, and $\mathcal{L} \subset \mathcal{K}$.*

An aggregator $\mathbb{A} : \mathrm{Seq}(\mathcal{K}) \to \mathcal{K}$ is $\mathcal{L}$-forcing if, for all sequences of knowledge operators $(K^1, \ldots, K^n) \in \mathrm{Seq}(\mathcal{K})$, the knowledge aggregation $\mathbb{A}(K^1, \ldots, K^n)$ is in $\mathcal{L}$.

Clearly if $\mathcal{L}$ is forced by some aggregator $\mathbb{A}$, then it is also preserved by $\mathbb{A}$. Proposition 11 contains a complete classification of the classical aggregators are either $\mathcal{L}$-preserving, $\mathcal{L}$-forcing, or neither, for the sets of knowledge operators satisfying each of the S5 axioms. While both distributed knowledge and common knowledge preserve the S5 axioms collectively, as in Proposition 7, only the distributed knowledge preserves the axioms individually. Surprisingly, the common knowledge aggregator does not force, or even preserve, the Positive Introspection property. This last fact is explored in greater detail in Chapter 2.

Proposition 11. *Table 1 indicates which properties are preserved or forced by the various aggregators. An entry "P" in the row of $\mathcal{L}$ and column of aggregator $\mathbb{A}$ indicates that $\mathbb{A}$ preserves $\mathcal{L}$. An entry "F" in such location indicates that $\mathbb{A}$ forces $\mathcal{L}$. An entry of "No" indicates that it does neither.*

	$\vee$	$\curlyvee$	$\curlywedge$	$\wedge$
$\mathcal{K}_0$	F	P	P	P
$\mathcal{K}_1$	F	P	P	P
$\mathcal{K}_2$	F	No	P	No
$\mathcal{K}_3$	P	P	P	P
$\mathcal{K}_4$	P	No	No	No
$\mathcal{K}_5$	P	No	No	No

Table 1: S5 Preservation

4.3 Monotonicity

The Monotonicity axiom, K.1., which states that whenever $E \subset F$, then $KE \subset KF$, can be considered to be a very reasonable assumption on the set of knowledge operators. All Monotonicity requires is that if an event E is a sub-event of F, then whenever the agent knows the sub-event E, they also know the super-event F. This axiom can be considered reasonable, but is also an extremely powerful assumption for producing more regular results. This short section assumes Monotonicity of all knowledge operators, in order to produce stronger results on the knowledge aggregators. Lemma 5 gives a more concise form of the distributed knowledge aggregator. Lemma 6 shows that the somebody knows aggregator preserves Positive Introspection. Proposition 12 shows that the common knowledge aggregator is increasing.

If all agents' knowledge operators satisfy Monotonicity, then the distributed knowledge aggregator $\vee$ admits a much cleaner representation.

Lemma 5. *Fix a state space Ω, and let $\mathcal{K}$ be the associated set of knowledge operators.*

Let $\vee : \mathrm{Seq}(\mathcal{K}) \to \mathcal{K}$ be the distributed knowledge aggregator given by Equation 1. Let $(K^1, \ldots, K^n) \in \mathrm{Seq}(\mathcal{K}_1)$ so that all operators satisfy Monotonicity. Then for all events

$E \in 2^\Omega$,

$$\vee (K^1, \ldots, K^n)E = \bigcap_{\bar{\omega} \notin E} \bigcup_{j \in J} K^j \neg\{\bar{\omega}\} \tag{7}$$

The somebody knows aggregator $\curlyvee$ does not, in general, preserve Positive Introspection. However, if all aggregators a Monotonic as well as satisfy Positive Introspection, then the somebody knows aggregation will also have Positive Introspection, as in Lemma 6. This is a useful result for proving that distributed knowledge preserves $\mathcal{K}_4$.

Lemma 6. *Fix a state space Ω, let $\mathcal{K}$ be the associated set of knowledge operators.*

If all operators are Monotonic, then the somebody knows aggregator $\curlyvee$ preserves axiom K.4. That is, for any sequence $(K^1, \ldots, K^n) \in \mathrm{Seq}(\mathcal{K}_1 \cap \mathcal{K}_4)$ then $\curlyvee(K^1, \ldots, K^n) \in \mathcal{K}_4$.

As noted in Section 3.2, the common knowledge aggregator is not, in general, an increasing aggregator. In the example given to show this, Example 4, the knowledge operator of the agent did not satisfy Monotonicity. This is indicative of a more general result. Indeed, if all operators are Monotonic, then the common knowledge aggregator is increasing, as in Proposition 12.

Proposition 12. *If all operators are Monotonic, then the common knowledge aggregator $\wedge$ is increasing.*

4.4 Cognitive Dissonance

Consider the following property of operators, which prevents behavior that could be thought of as a form of cognitive dissonance. In the Modal Logic literature this is called Axiom D.

Definition 24. *Fix a state space Ω, and let $\mathcal{K}$ be the associated set of knowledge operators.*

An operator $K \in \mathcal{K}$ satisfies Axiom D if for all $E \in 2^\Omega$, then $KE \cap K\neg E = \emptyset$. The set of operators satisfying Axiom D is denoted $\mathcal{K}_D$.

$$\mathcal{K}_D = \{K \in \mathcal{K} \mid \; KE \cap K\neg E = \emptyset \textit{ for all } E \in 2^\Omega\}$$

An operator satisfying Axiom D is described as being non-dissonant.

An operator satisfies Axiom D if there is no state at which the agent knows that some event E both has, and has not, occurred. It is immediate that the axiom of Truth implies Axiom D, $\mathcal{K}_3 \subset \mathcal{K}_D$. Moreover, if $K\emptyset = \emptyset$ and $K \in \mathcal{K}_2$, then $K \in \mathcal{K}_D$ because $KE \cap K\neg E \subset K(E \cap \neg E) = K\emptyset = \emptyset$.

Combined, these two properties reveal that Axiom D is a mild requirement on operators. Intuitively speaking, operators which are not non-dissonant are relatively far from partitional operators.

Definition 25. *Fix a state space Ω, and let $\mathcal{K}$ be the associated set of knowledge operators. An aggregator $\mathbb{A} : \mathrm{Seq}(\mathcal{K}) \to \mathcal{K}$ preserves Axiom D if for any sequence $(K^1, \ldots, K^n) \in \mathrm{Seq}(\mathcal{K})$ such that each $K^i \in \mathcal{K}_D$, then $\mathbb{A}(K^1, \ldots, K^n) \in \mathcal{K}_D$.*

Every negative aggregator $\mathbb{A} : \mathrm{Seq}(\mathcal{K}) \to \mathcal{K}$ preserves Axiom D. This is fairly immediate as negative aggregators shrink what is known, and if less is known, then it is easier for Axiom D to be satisfied. Moreover, if at least one agent being aggregated by a negative aggregator satisfies Axiom D, then the resulting aggregation will also satisfy Axiom D. Common knowledge and everybody knows is particular preserve Axiom D.

Proposition 13. *Fix a state space Ω, and let $\mathcal{K}$ be the associated set of knowledge operators. Let $\mathbb{A} : \mathrm{Seq}(\mathcal{K}) \to \mathcal{K}$ be a negative aggregator. Then, $\mathbb{A}$ preserves Axiom D. Moreover, for $(K^1, \ldots, K^n) \in \mathrm{Seq}(\mathcal{K})$ such that $K^i \in \mathcal{K}_D$ for at least one $i = 1, \ldots, n$, then $\mathbb{A}(K^1, \ldots, K^n)$ satisfies Axiom D.*

However, in the absence of other properties, even the minimally positive aggregator, somebody knows, does not preserve non-dissonance, as in Example 7. Moreover, every positive aggregator does not preserve non-dissonance, as in Proposition 14.

Example 7. *Let $\Omega = \{a, b\}$, $J = \{1, 2\}$. Define $K^1E = E$, for every event $E \subset \Omega$. Let $K^2\{a\} = \{b\}$, $K^2\{b\} = \{a\}$, $K^2\emptyset = \emptyset$, and $K^2\Omega = \Omega$. In this case, $K^1 \in \mathcal{K}_D$, $K^2 \in \mathcal{K}_D$, but $\curlyvee^J\{a\} = \curlyvee^J\{b\} = \Omega$, violating Axiom D.*

Proposition 14. *Fix a state space Ω, with $|\Omega| \geq 2$, and let $\mathcal{K}$ be the associated set of knowledge operators.*

Let $\mathbb{A} : \mathrm{Seq}(\mathcal{K}) \to \mathcal{K}$ be a positive aggregator. Aggregator $\mathbb{A}$ does not preserve Axiom D.

5 Knowledge Modeling through Collections of Events

So far we have been working with knowledge operators $K : 2^\Omega \to 2^\Omega$ with the interpretation that for each event E, that KE is the set of states where the agent knows E has occurred. An alternative formalism is to take a function $\mathcal{E} : \Omega \to 2^{2^\Omega}$ mapping states to collections of events. The interpretation is that $\mathcal{E}(\omega)$ contains those events the agent considers possible when the true state is ω. Equation 8 formalizes the link between knowledge represented as collections of known events $\mathcal{E}$, and knowledge represented using a knowledge operator K. Proposition 15 shows that these formulations are equivalent. This alternative approach gives two main advantages, i) for proofs it is occasionally more useful to work in terms of collections of known events, and ii) for intuition and interpretation it can be helpful to have another way of thinking about the problem.

Fix a non-empty, finite state space Ω. Consider a knowledge operator $K : 2^\Omega \to 2^\Omega$. For each $\omega \in \Omega$, define the collection of all *known* events at ω, denoted $\mathcal{E}_K(\omega)$, by

$$\mathcal{E}_K(\omega) = \{E \in 2^\Omega \mid \omega \in KE\} \tag{8}$$

The collection $\mathcal{E}_K(\omega)$ may be empty. However, if $K\Omega = \Omega$, then $\Omega \in \mathcal{E}_K(\omega)$ because $\omega \in \Omega = K\Omega$. In particular, if K is a correspondence operator, by Lemma 3, $K\Omega = \Omega$, so $\Omega \in \mathcal{E}_K(\omega)$ for all $\omega \in \Omega$.

Moreover, suppose $K : 2^\Omega \to 2^\Omega$ is a correspondence operator represented by the correspondence $\gamma : \Omega \to 2^\Omega$, so that $K = K_\gamma$ in the sense of Definition 20. Then the collection of known events function $\mathcal{E}_{K_\gamma}$ takes the form

$$\mathcal{E}_{K_\gamma}(\omega) = \{E \in 2^\Omega \mid \omega \in K_\gamma E\} = \{E \in 2^\Omega \mid \gamma(\omega) \subset E\}$$

Equation 8 describes how to transform a knowledge operator $K : 2^\Omega \to 2^\Omega$ into a collection of events for each state. This is a mapping from knowledge operators to functions taking each state $\omega \in \Omega$ into a collection of events where $E \in \mathcal{E}_K(\omega)$ if and only if $\omega \in KE$.

In the opposite direction, consider any arbitrary function $\mathcal{E} : \Omega \to 2^{2^\Omega}$ taking states into collections of events. From this function it is possible to define a knowledge operator $K_\mathcal{E} : 2^\Omega \to 2^\Omega$, as follows. For every event $E \in 2^\Omega$, define $K_\mathcal{E} E$ by:

$$K_\mathcal{E} E = \{\omega \in \Omega \mid E \in \mathcal{E}(\omega)\} \tag{9}$$

This means that $E \in \mathcal{E}(\omega)$ if and only if $\omega \in K_\mathcal{E} E$. Moreover, these transformations are inverse functions, as shown in Proposition 15.

Proposition 15. *Fix a state space Ω and let $\mathcal{K}$ be the associated set of knowledge operators.*

Consider the mapping taking each knowledge operator $K : 2^\Omega \to 2^\Omega$ into a function $\mathcal{E}_K : \Omega \to 2^{2^\Omega}$, according to Equation 8. Consider another mapping taking a function $\mathcal{E} : \Omega \to 2^{2^\Omega}$ into a knowledge operator $K_\mathcal{E} : 2^\Omega \to 2^\Omega$, according to Equation 9. These two mappings are inverse functions. That is, for every function $\mathcal{E} : \Omega \to 2^{2^\Omega}$ and state $\omega \in \Omega$:

$$\mathcal{E}_{K_\mathcal{E}}(\omega) = \mathcal{E}(\omega)$$

and for every knowledge operator $K \in \mathcal{K}$ and event $E \in 2^\Omega$:

$$K_{\mathcal{E}_K} E = KE$$

The next result explains how these mappings act on Boolean set operations.

Proposition 16. *Suppose that $\circledast$ represents the Boolean set operation of union or intersection. Consider two knowledge operators $K^1 : 2^\Omega \to 2^\Omega$ and $K^2 : 2^\Omega \to 2^\Omega$. Define the knowledge operator $K^1 \circledast K^2 : 2^\Omega \to 2^\Omega$, at each $E \in 2^\Omega$ by:*

$$(K^1 \circledast K^2)E = K^1 E \circledast K^2 E$$

The mapping taking knowledge operators $K : 2^\Omega \to 2^\Omega$ into functions $\mathcal{E}_K : \Omega \to 2^{2^\Omega}$ preserves the set operation $\circledast$. Formally, for every pair of knowledge operators $K^1 : 2^\Omega \to 2^\Omega$ and $K^2 : 2^\Omega \to 2^\Omega$, and for every $\omega \in \Omega$:

$$\mathcal{E}_{K^1 \circledast K^2}(\omega) = \mathcal{E}_{K^1}(\omega) \circledast \mathcal{E}_{K^2}(\omega)$$

Consider a knowledge aggregator $\mathbb{A} : \mathrm{Seq}(\mathcal{K}) \to \mathcal{K}$. Exploring the bijection between knowledge operators $K : 2^\Omega \to 2^\Omega$ and functions $\mathcal{E} : \Omega \to 2^{2^\Omega}$, it is possible to define aggregators by establishing how they act on all finite sequences of functions $\mathcal{E} : \Omega \to 2^{2^\Omega}$.

Let $\overline{\mathcal{E}}$ be the set of all functions $\mathcal{E} : \Omega \to 2^{2^\Omega}$. Let $\mathrm{Seq}(\overline{\mathcal{E}})$ be the set of finite sequences of functions $\mathcal{E} : \Omega \to 2^{2^\Omega}$. A C-aggregator is a function $\mathcal{A} : \mathrm{Seq}(\overline{\mathcal{E}}) \to \overline{\mathcal{E}}$. The letter C stands for "collection".

Given an aggregator $\mathbb{A} : \mathrm{Seq}(\mathcal{K}) \to \mathcal{K}$, it is possible to defined its associated C-aggregator as follows. Consider any finite sequence of functions $\mathcal{E}_j : \Omega \to 2^{2^\Omega}$, where $j \in J = \{1, 2, \cdots, |J|\}$. Define knowledge operators $K^j : 2^\Omega \to 2^\Omega$, for each $j \in J$, pointwise, via Equation 9, by:

$$K^j E = K_{\mathcal{E}_j} E = \{\omega \in \Omega \mid E \in \mathcal{E}_j(\omega)\}$$

From the sequence $(K^1, \cdots, K^{|J|})$, compute the knowledge aggregation $\mathbb{A}(K^1, \cdots, K^{|J|})$. Applying Equation 8 to $\mathbb{A}(K^1, \cdots, K^{|J|})$ gives

$$\mathcal{A}(\mathcal{E}_1, \cdots, \mathcal{E}_{|J|}) = \mathcal{E}_{\mathbb{A}(K^1, \cdots, K^{|J|})} = \mathcal{E}_{\mathbb{A}\left(K_{\mathcal{E}_1}, \cdots, K_{\mathcal{E}_{|J|}}\right)}$$

where, for each state $\omega \in \Omega$,

$$\mathcal{E}_{\mathbb{A}(K^1, \cdots, K^{|J|})}(\omega) = \{E \subset \Omega \mid \omega \in \mathbb{A}(K^1, \cdots, K^{|J|})E\}$$

Another way to understand this process is the following. Fix any arbitrary finite sequence of knowledge operators $(K^1, \cdots, K^{|J|})$, where $K^j : 2^\Omega \to 2^\Omega$, for all $j \in J$. First, apply the aggregator $\mathbb{A} : \mathrm{Seq}(\mathcal{K}) \to \mathcal{K}$ to the sequence $(K^1, \cdots, K^{|J|})$. Thus, $\mathbb{A}(K^1, \cdots, K^{|J|}) : 2^\Omega \to 2^\Omega$ is a knowledge operator. Then apply Equation 8 to $\mathbb{A}(K^1, \cdots, K^{|J|}) : 2^\Omega \to 2^\Omega$ to find the function $\mathcal{E}_{\mathbb{A}(K^1, \cdots, K^{|J|})} : \Omega \to 2^{2^\Omega}$.

Alternatively, start with the same sequence of knowledge operators $(K^1, \cdots, K^{|J|})$ and apply the Equation 8 to each $K^j : 2^\Omega \to 2^\Omega$, with $j \in J$. This leads to the sequence $(\mathcal{E}_1, \cdots, \mathcal{E}_{|J|}) \in \mathrm{Seq}(\overline{\mathcal{E}})$, where $\mathcal{E}_j(\omega) = \mathcal{E}_{K^j}(\omega)$, for all $\omega \in \Omega$. Then take the sequence $(\mathcal{E}_1, \cdots, \mathcal{E}_{|J|})$ and apply the C-aggregator $\mathcal{A} : \mathrm{Seq}(\overline{\mathcal{E}}) \to \overline{\mathcal{E}}$. The result is the same function from Ω to 2^{2^Ω}. That is

$$\mathcal{A}(\mathcal{E}_1, \cdots, \mathcal{E}_m) = \mathcal{E}_{\mathbb{A}(K^1, \cdots, K^m)}$$

The C-aggregator $\mathcal{A}$ is defined so that Figure 1 commutes. Let $f : \mathcal{K} \to \bar{\mathcal{E}}$ be the mapping given by Equation 8. Let $\operatorname{Seq}(f) : \operatorname{Seq}(\mathcal{K}) \to \operatorname{Seq}(\bar{\mathcal{E}})$ be the mapping such that $\operatorname{Seq}(f)(K^1, \ldots, K^n) = (f(K^1), \ldots, f(K^n))$.

$$\begin{tikzcd}
\operatorname{Seq}(\mathcal{K}) \arrow[r, "\mathbb{A}"] \arrow[d, "\operatorname{Seq}(f)"'] & \mathcal{K} \arrow[d, "f"] \\
\operatorname{Seq}(\bar{\mathcal{E}}) \arrow[r, "\mathcal{A}"'] & \bar{\mathcal{E}}
\end{tikzcd}$$

Figure 1: Aggregation commutes

Such a C-aggregator which makes Figure 1 commute certainly exists, as, by Proposition 15, the mappings f and therefore $\operatorname{Seq}(f)$, are bijective.

In some cases the C-aggregator corresponding to a particular knowledge aggregator has an explicit formulation. In particular, Lemma 7 gives an explicit form of the C-aggregator corresponding to distributed knowledge.

Consider a knowledge model $(\Omega, J, (K^j)_{j\in J})$, and fix any $\omega \in \Omega$ and $j \in J$. Let $E^j_*(\omega)$ be the intersection over all events in $\mathcal{E}_{K^j}(\omega)$. Formally

$$E^j_*(\omega) = \bigcap_{E \in \mathcal{E}_{K^j}(\omega)} E \tag{10}$$

If ω is the true state, and operator K^j satisfies Conjunction, the event $E^j_*(\omega)$ would be the set of states agent j considers possible when the true state is ω. Similarly, let E^J_* be the intersection over all events in $\mathcal{E}_{K^j}(\omega)$ for all $j \in J$. Formally

$$E^J_*(\omega) = \bigcap_{j\in J} E^j_* = \bigcap_{j\in J} \bigcap_{E \in \mathcal{E}_{K^j}(\omega)} E \tag{11}$$

If ω is the true state, and all agents' operators satisfy Conjunction, the event $E^J_*(\omega)$ would be the set of states which all agents consider possible.

Lemma 7. *Let $(\Omega, J, (K^j)_{j\in J})$ be a knowledge model. The collection of known events for the distributed knowledge aggregation $\mathcal{E}_{\vee^J}(\omega)$ is the collection of supersets of $E_*^J(\omega)$. That is,*

$$\mathcal{E}_{\vee^J}(\omega) = \{E \in 2^\Omega \mid E \supset E_*^J(\omega)\} \tag{12}$$

Example 8 illustrates the preceding discussion.

Example 8. *Let $(\Omega, J, (K^j)_{j\in J})$ be a knowledge model, with $\Omega = \{a, b, c\}$ and $J = \{1, 2\}$. Let the knowledge operators K^1, $K^2 : 2^\Omega \to 2^\Omega$ be defined by*

E	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	Ω
K^1E	$\emptyset$	$\{a\}$	$\{b,c\}$	$\{c\}$	$\{a\}$	$\{a\}$	$\{b\}$	Ω
K^2E	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	Ω

The collections of known events $\mathcal{E}_{K^1}$ and $\mathcal{E}_{K^2}$ are then

ω	a	b	c
$\mathcal{E}_{K^1}(\omega)$	$\{\Omega, \{a\}, \{a,b\}, \{a,c\}\}$	$\{\Omega, \{b\}, \{b,c\}\}$	$\{\Omega, \{b\}, \{c\}\}$
$\mathcal{E}_{K^2}(\omega)$	$\{\Omega, \{a,b\}, \{a,c\}\}$	$\{\Omega, \{a,b\}, \{b,c\}\}$	$\{\Omega, \{a,c\}, \{b,c\}\}$

While here we constructed $\mathcal{E}_{K^1}$ from K^1, it is straightforward to check that if we apply Equation 9 to $\mathcal{E}_{K^1}$, then we return to K^1. Similarly for $\mathcal{E}_{K^2}$ and K^2.

We can also use Lemma 7 to find the C-aggregator associated with the distributed knowledge aggregator. The values of E_*^j and E_*^J, and the resulting function $\mathcal{E}_{\vee^J}$ of Equation 12 are

ω	a	b	c
$E_*^1(\omega)$	$\{a\}$	$\{b\}$	$\emptyset$
$E_*^2(\omega)$	$\{a\}$	$\{b\}$	$\{c\}$
$E_*^J(\omega)$	$\{a\}$	$\{b\}$	$\emptyset$
$\mathcal{E}_{\vee^J}(\omega)$	$\{\Omega, \{a\}, \{a,b\}, \{a,c\}\}$	$\{\Omega, \{b\}, \{a,b\}, \{b,c\}\}$	2^Ω

A Proofs

This appendix contains the remaining proofs. Many proofs presented rely on the notation, and sometimes results, from Section 5. Readers are encouraged to read Section 5 prior to reading the proofs.

In most cases the proofs are presented in the order they appear in the main text. When proofs rely on propositions stated later in the text, the proof has been moved so that each proof relies only on proofs already given. In particular, Proposition 7 follows Proposition 8, which follows Proposition 11. Lemma 3 follows Proposition 9, and Lemma 6 precedes Proposition 11. All results of Section 5, namely Propositions 15 and 16 and Lemma 7, are moved to the front.

Proof of Proposition 15:

First show $\mathcal{E}_{K_\mathcal{E}} = \mathcal{E}$ for all $\mathcal{E} : \Omega \to 2^{2^\Omega}$. Fix a function $\mathcal{E} : \Omega \to 2^{2^\Omega}$ and state $\omega \in \Omega$. By Equation 9, the operator $K_\mathcal{E} \in \mathcal{K}$ is given by

$$K_\mathcal{E} E = \{\omega \in \Omega \mid E \in \mathcal{E}(\omega)\}$$

By Equation 8, the function $\mathcal{E}_{K_\mathcal{E}} : \Omega \to 2^{2^\Omega}$ is given by

$$\mathcal{E}_{K_\mathcal{E}}(\omega) = \{E \in 2^\Omega \mid \omega \in K_\mathcal{E} E\} = \{E \in 2^\Omega \mid E \in \mathcal{E}(\omega)\} = \mathcal{E}(\omega)$$

As this is true for all $\omega \in \Omega$, then $\mathcal{E}_{K_\mathcal{E}} = \mathcal{E}$.

Similarly, show $K_{\mathcal{E}_K} = K$ for all operators $K \in \mathcal{K}$. Fix an operator $K \in \mathcal{K}$ and event $E \in 2^\Omega$. By Equation 8, the function $\mathcal{E} : \Omega \to 2^{2^\Omega}$ is given by

$$\mathcal{E}_K(\omega) = \{E \in 2^\Omega \mid \omega \in KE\}$$

By Equation 9, the operator $K_{\mathcal{E}_K} \in \mathcal{K}$ is given by

$$K_{\mathcal{E}_K} E = \{\omega \in \Omega \mid E \in \mathcal{E}_K(\omega)\} = \{\omega \in \Omega \mid \omega \in KE\} = KE$$

As this is true for all $E \in 2^\Omega$, then $K_{\mathcal{E}_K} = K$, as required. □

Proof of Proposition 16:

By definition, the following statements are equivalent:

$$
\begin{aligned}
& E \in \mathcal{E}_{K^1 \circledast K^2}(\omega) \\
\iff\ & \omega \in (K^1 \circledast K^2)E \\
\iff\ & \omega \in K^1 E \circledast K^2 E.
\end{aligned}
$$

If $\circledast = \cup$ (the union), the last statement means that $\omega \in K^1 E \cup K^2 E$, so $\omega \in K^1 E$ or $\omega \in K^2 E$. If $\circledast = \cap$ (the intersection), the last statement means that $\omega \in K^1 E \cap K^2 E$, consequently $\omega \in K^1 E$ and $\omega \in K^2 E$. But $\omega \in K^1 E$ if and only if $E \in \mathcal{E}_{K^1}(\omega)$. Also, $\omega \in K^2 E$ if and only if $E \in \mathcal{E}_{K^2}(\omega)$. Hence, $E \in \mathcal{E}_{K^1}(\omega) \circledast \mathcal{E}_{K^2}(\omega)$. This proves $E \in \mathcal{E}_{K^1 \circledast K^2}(\omega)$ if and only if $E \in \mathcal{E}_{K^1}(\omega) \circledast \mathcal{E}_{K^2}(\omega)$. $\square$

Proof of Lemma 7:

Let $(\Omega, J, (K^j)_{j \in J})$ be a knowledge model, and $\omega \in \Omega$. The set $\mathcal{E}_{\vee^J}(\omega)$ is given by Equation 8 as

$$\mathcal{E}_{\vee^J}(\omega) = \{E \in 2^\Omega \mid \omega \in \vee^J E\}$$

Let $\hat{\mathcal{E}}_{\vee^J}(\omega)$ be the set given in Equation 12. That is

$$\hat{\mathcal{E}}_{(}\omega) = \{E \in 2^\Omega \mid E \supset E_*^J(\omega)\}$$

The content of the lemma is to show that $\mathcal{E}_{\vee^J}(\omega) = \hat{\mathcal{E}}(\omega)$. It will be enough to show that $E \in \mathcal{E}_{\vee^J}(\omega)$ if and only if $E \in \hat{\mathcal{E}}(\omega)$. That is, we want to show

$$\omega \in \vee^J E \iff E \supset E_*^J(\omega)$$

where $E_*^J(\omega)$ is defined in Equations 10 and 11. Expanding the definition of $\vee^J$ from Equation 1, and writing in terms of existential quantifiers gives,

$$
\begin{aligned}
& \omega \in \vee(K^1, \ldots, K^n)E \\
\iff & \omega \in \bigcap_{\bar{\omega} \notin E} \bigcup_{F \subset \neg\{\bar{\omega}\}} \bigcup_{j \in J} K^j F \\
\iff & \left[\forall \bar{\omega} \notin E,\ \exists j \in J, F \subset \neg\{\bar{\omega}\} \text{ such that } \omega \in K^j F\right]
\end{aligned}
$$

Using the definition that $\omega \in K^j F$ if and only if $F \in \mathcal{E}_{K^j}(\omega)$, and rearranging conditions yields,

$$\iff [\forall \bar{\omega} \notin E,\ \exists j \in J, F \subset \neg\{\bar{\omega}\} \text{ such that } F \in \mathcal{E}_{K^j}(\omega)]$$
$$\iff [\forall \bar{\omega} \notin E,\ \exists j \in J, F \in \mathcal{E}_{K^j}(\omega) \text{ such that } \bar{\omega} \notin F]$$

Rearranging the logic and taking the contrapositive produces,

$$\iff [(\bar{\omega} \notin E) \implies (\exists j \in J, F \in \mathcal{E}_{K^j}(\omega) \text{ such that } \bar{\omega} \notin F)]$$
$$\iff [(\forall j \in J, F \in \mathcal{E}_{K^j}(\omega) \text{ with } \bar{\omega} \in F) \implies (\bar{\omega} \in E)]$$

and putting back in set notation obtains,

$$\iff \left[\bar{\omega} \in \bigcap_{j \in J} \bigcap_{F \in \mathcal{E}_{K^j}(\omega)} F \implies \bar{\omega} \in E\right]$$
$$\iff \left[\bigcap_{j \in J} \bigcap_{F \in \mathcal{E}_{K^j}(\omega)} F \subset E\right]$$
$$\iff E \supset E_*^J(\omega)$$

as required. $\square$

Proof of Proposition 1:

Suppose aggregator $\mathbb{A}$ satisfies Equation 3. Let $x, y, z \in \mathrm{Seq}(\mathcal{K})$ be sequences of knowledge operators. As $\mathbb{A}$ satisfies Equation 3, $\mathbb{A}(x, \mathbb{A}(y), z) = \mathbb{A}(\mathbb{A}(x), \mathbb{A}(\mathbb{A}(y)), \mathbb{A}(z))$. If $\mathbb{A}$ satisfies Equation 3, then for $n_1 = 1$ and $n_2 = n$, we have $\mathbb{A}(K^1, \ldots, K^n) = \mathbb{A}(\mathbb{A}(K^1, \ldots, K^n))$. That is, the aggregator $\mathbb{A}$ is idempotent. In particular, $\mathbb{A}(\mathbb{A}(y)) = \mathbb{A}(y)$ as $y \in \mathrm{Seq}(\mathcal{K})$. Therefore $\mathbb{A}(\mathbb{A}(x), \mathbb{A}(\mathbb{A}(y)), \mathbb{A}(z)) = \mathbb{A}(\mathbb{A}(x), \mathbb{A}(y), \mathbb{A}(z))$. Applying Equation 3 again gives

$$\mathbb{A}(\mathbb{A}(x), \mathbb{A}(y), \mathbb{A}(z)) = \mathbb{A}(x, y, z)$$

Therefore $\mathbb{A}(x, y, z) = \mathbb{A}(x, \mathbb{A}(y), z)$ as required.

Suppose aggregator $\mathbb{A}$ satisfies Equation 2. Let $(K^1, \ldots, K^n) \in \mathrm{Seq}(\mathcal{K})$ be a sequence of knowledge operators, and $0 = n_0 < n_1 < \cdots < n_k = n$. As $\mathbb{A}$ satisfies Equation 2, for

$x = (K^1, \dots, K^{n_1})$, $y = (K^{n_1+1}, \dots, K^{n_2})$, and $z = (K^{n_2+1}, \dots, K^n)$, then

$$\mathbb{A}(K^1, \dots, K^n) = \mathbb{A}(K^1, \dots, K^{n_1}, \mathbb{A}(K^{n_1+1}, \dots, K^{n_2}), K^{n_2+1}, \dots, K^n)$$

Similarly, for

$$x = (K^1, \dots, K^{n_1}, \mathbb{A}(K^{n_1+1}, \dots, K^{n_2})),\ y = (K^{n_2+1}, \dots, K^{n_3}), \text{ and } z = (K^{n_3+1}, \dots, K^n)$$

then

$$\mathbb{A}(K^1, \dots, K^n) = \mathbb{A}(K^1, \dots, K^{n_1}, \mathbb{A}(K^{n_1+1}, \dots, K^{n_2}), \mathbb{A}(K^{n_2+1}, \dots, K^{n_3}), K^{n_3+1}, \dots, K^n)$$

Continuing in this manner we have

$$\mathbb{A}(K^1, \dots, K^n) = \mathbb{A}(K^1, \dots, K^{n_1}, \mathbb{A}(K^{n_1+1}, \dots, K^{n_2}), \dots, \mathbb{A}(K^{n_{k-1}+1}, \dots, K^n))$$

Finally for x is the empty list,

$$y = (K^1, \dots, K^{n_1}), \text{ and } z = (A(K^{n_1+1}, \dots, K^{n_2}), \dots, \mathbb{A}(K^{n_{k-1}+1}, \dots, K^n))$$

then

$$\mathbb{A}(K^1, \dots, K^n) = \mathbb{A}(\mathbb{A}(K^1, \dots, K^{n_1}), \mathbb{A}(K^{n_1+1}, \dots, K^{n_2}), \dots, \mathbb{A}(K^{n_{k-1}+1}, \dots, K^n))$$

so $\mathbb{A}$ satisfies Equation 3, as required. □

Proof of Proposition 2:

Let $J \in \text{Seq}(\mathcal{K})$, and $J = J_1 \circ \dots \circ J_n$ be a breakdown of the agents into n shorter lists $J_1, \dots, J_n$.

The **everybody knows** aggregator $\curlywedge$ is associative as

$$\begin{aligned} \curlywedge^J E &= \bigcap_{j \in J} K^j E \\ &= \bigcap_{i=1}^{n} \bigcap_{j \in J_i} K^j E \\ &= \bigcap_{i=1}^{n} \curlywedge^{J_i} E = \curlywedge(\curlywedge^{J_1}, \dots, \curlywedge^{J_n}) E \end{aligned}$$

Similarly, the **somebody knows** aggregator $\curlyvee$ is associative as

$$\begin{aligned}\curlyvee^J E &= \bigcup_{j\in J} K^j E \\ &= \bigcup_{i=1}^{n} \bigcup_{j\in J_i} K^j E \\ &= \bigcup_{i=1}^{n} \curlyvee^{J_i} E = \curlyvee(\curlyvee^{J_1}, \dots, \curlyvee^{J_n}) E\end{aligned}$$

Showing the **distributed knowledge** aggregator $\vee$ is associative is somewhat more involved. The notation and results of Section 5 are used for this proof. As the mapping given by Equation 8 is a bijection, as noted in Proposition 15, to show that $\vee$ is associative it is sufficient to show that

$$\mathcal{E}_{\vee^J} = \mathcal{E}_{\vee(\vee^{J_1},\dots,\vee^{J_n}))}$$

Fix $\omega \in \Omega$. By Lemma 7,

$$\mathcal{E}_{\vee(K^1,\dots,K^n)}(\omega) = \{E \in 2^\Omega \mid E \supset E_*^J(\omega)\}$$

where

$$E_*^J(\omega) = \bigcap_{j\in J} \bigcap_{F\in\mathcal{E}_{K^j}(\omega)} F$$

Similarly,

$$\mathcal{E}_{\vee(\vee^{J_1}),\dots,\vee^{J_n}))}(\omega) = \{E \in 2^\Omega \mid E \supset E_{**}(\omega)\}$$

where

$$E_{**}(\omega) = \bigcap_{i=1}^{n} \bigcap_{F\in\mathcal{E}_{\vee^{J_i}}(\omega)} F$$

In particular, if $E_*^J = E_{**}$, then $\mathcal{E}_{\vee(K^1,\dots,K^n)}(\omega) = \mathcal{E}_{\vee(\vee^{J_1}),\dots,\vee^{J_n}))}(\omega)$.

From Equation 12, $\mathcal{E}_{\vee^{J_i}}(\omega) = \{E \in 2^\Omega \mid E \supset E_*^{J_i}(\omega)\}$ for each J_i, where $E_*^{J_i}(\omega) = \bigcap_{j\in J_i} E_*^j(\omega)$. Therefore

$$E_{**}(\omega) = \bigcap_{i=1}^{n} \bigcap_{F\in\mathcal{E}_{\vee^{J_i}}(\omega)} F = \bigcap_{i=1}^{n} \bigcap_{F\mid F\supset E_*^{J_i}(\omega)} F$$

As the intersection over the supersets of some set X is just the set X itself,

$$\bigcap_{i=1}^{n} \bigcap_{F \mid F \supset E_*^{J_i}(\omega)} F = \bigcap_{i=1}^{n} E_*^{J_i}$$

Substituting the definition of $E_*^{J_i}(\omega)$ gives

$$\bigcap_{i=1}^{n} E_*^{J_i} = \bigcap_{i=1}^{n} \bigcap_{j \in J_i} \bigcap_{F \in \mathcal{E}_{K^j}(\omega)} F$$

Finally as $J_1, \ldots, J_n$ partition the list J,

$$\bigcap_{i=1}^{n} \bigcap_{j \in J_i} \bigcap_{F \in \mathcal{E}_{K^j}(\omega)} F = \bigcap_{j \in J} \bigcap_{F \in \mathcal{E}_{K^j}(\omega)} F = E_*^{J}(\omega)$$

which is to say

$$E_{**}(\omega) = E_*^{J}(\omega)$$

as required. □

Proof of Proposition 3:

Let $(\Omega, J, \{K^j\}_{j \in J}\}$ be a knowledge model. Let $f : \Omega \to \Omega$ be a bijection. For an event $E \in 2^{\Omega}$, we abuse notation and write

$$f(E) = \{f(\omega) \in \Omega \mid \omega \in E\}$$

and similarly for f^{-1}. This means $f^{-1}(f(E)) = f(f^{-1}(E)) = E$ for all $E \in 2^{\Omega}$. As f is a bijection on Ω, it distributes over unions and intersections in the sense that $f(E \cup F) = f(E) \cup f(F)$ and $f(E \cap F) = f(E) \cap f(F)$ for all $E, F \in 2^{\Omega}$. The ability to move the function f freely across unions and intersections drives the remainder of the proof.

Consider the **somebody knows** aggregator $\curlyvee$. For any $E \in 2^\Omega$,

$$\begin{aligned} f^{-1} \circ \left[\curlyvee \left(f \circ K^1, \ldots, f \circ K^n\right)\right] E &= f^{-1}\left(\bigcup_{j \in J} f(K^j E)\right) \\ &= f^{-1}\left(f\left(\bigcup_{j \in J} K^j E\right)\right) \\ &= \bigcup_{j \in J} K^j E = \curlyvee(K^1, \ldots, K^n)E \end{aligned}$$

As this hold for all $E \in 2^\Omega$, the somebody knows aggregator is label neutral.

Next, consider the **everybody knows** aggregator $\curlywedge$. For any $E \in 2^\Omega$,

$$\begin{aligned} f^{-1} \circ \left[\curlywedge \left(f \circ K^1, \ldots, f \circ K^n\right)\right] E &= f^{-1}\left(\bigcap_{j \in J} f(K^j E)\right) \\ &= f^{-1}\left(f\left(\bigcap_{j \in J} K^j E\right)\right) \\ &= \bigcap_{j \in J} K^j E = \curlywedge(K^1, \ldots, K^n)E \end{aligned}$$

As this hold for all $E \in 2^\Omega$, the everybody knows aggregator is label neutral.

Next, consider the **distributed knowledge** aggregator $\vee$. For any $E \in 2^\Omega$,

$$\begin{aligned} f^{-1} \circ \left[\vee \left(f \circ K^1, \ldots, f \circ K^n\right)\right] E &= f^{-1}\left(\bigcap_{\bar{\omega} \notin E} \bigcup_{F \subset \neg\{\bar{\omega}\}} \bigcup_{j \in J} f(K^j F)\right) \\ &= f^{-1}\left(f\left(\bigcap_{\bar{\omega} \notin E} \bigcup_{F \subset \neg\{\bar{\omega}\}} \bigcup_{j \in J} K^j F\right)\right) \\ &= \bigcap_{\bar{\omega} \notin E} \bigcup_{F \subset \neg\{\bar{\omega}\}} \bigcup_{j \in J} K^j F \\ &= \vee(K^1, \ldots, K^n)E \end{aligned}$$

As this hold for all $E \in 2^\Omega$, the distributed knowledge aggregator is label neutral. □

Proof of Lemma 1:

First we show $\wedge^J E \subset \curlywedge^J E$.

$$\wedge^J E = \bigcap_{s=1}^{\infty} (\curlywedge^J)^{(s)} E = \curlywedge^J E \cap \bigcap_{s=2}^{\infty} (\curlywedge^J)^{(s)} E \subset \curlywedge^J E.$$

Next, we show $\curlywedge^J E \subset K^j E$ for each $j \in J$. Fix $i \in J$, then

$$\curlywedge^J E = \bigcap_{j \in J} K^j E \subset K^i E.$$

Third, we show $K^j E \subset \curlyvee^J E$ for all $j \in J$. Again, fix $i \in J$, then

$$K^i E \subset \bigcup_{j \in J} K^j E = \curlyvee^J E.$$

Finally, we show $\curlyvee^J E \subset \vee^J E$. Let $\omega \in \curlyvee^J E = \bigcup_{j \in J} K^j E$. Fix a state $\bar{\omega} \notin E$. As $E \subset \neg\{\bar{\omega}\}$, therefore $\omega \in \bigcup_{F \subset \neg\{\bar{\omega}\}} \bigcup_{j \in J} K^j F$. Since this is true for each $\bar{\omega} \notin E$, we have that

$$\omega \in \bigcap_{\bar{\omega} \notin E} \bigcup_{F \subset \neg\{\bar{\omega}\}} \bigcup_{j \in J} K^j F$$

and so, $\curlyvee^J E \subset \vee^J E$. $\square$

Proof of Proposition 4:

Let $\mathbb{A} : \mathrm{Seq}(\mathcal{K}) \to \mathcal{K}$ be an aggregator. Fix $J \in \mathrm{Seq}(\mathcal{K})$ and $E \in 2^\Omega$. Suppose $A^J E \supset \curlyvee^J E = \bigcup_{j \in J} K^j E$. As $A^J E$ is a superset of the union over all $K^j E$, it is a superset of each $K^j E$. As this is true for all $J \in \mathrm{Seq}(\mathcal{K})$ and $E \in 2^\Omega$, therefore $\mathbb{A}$ is positive. Similarly, fix $J \in \mathrm{Seq}(\mathcal{K})$ and $E \in 2^\Omega$ and suppose $\mathbb{A}$ is positive, so that $A^J E \supset K^j E$ for each agent $j \in J$. Then certainly $A^J E \supset \bigcup_{j \in J} K^j E = \curlyvee^J E$. Overall, $\mathbb{A}$ is positive if and only if A^J is more informative than $\curlyvee^J$.

Let $\mathbb{A} : \mathrm{Seq}(\mathcal{K}) \to \mathcal{K}$ be an aggregator. Fix $J \in \mathrm{Seq}(\mathcal{K})$ and $E \in 2^\Omega$. Suppose $A^J E \subset \curlywedge^J E = \bigcap_{j \in J} K^j E$. As $A^J E$ is a subset of the intersection over all $K^j E$, it is a subset of each $K^j E$. As this is true for all $J \in \mathrm{Seq}(\mathcal{K})$ and $E \in 2^\Omega$, therefore $\mathbb{A}$ is negative. Similarly, fix $J \in \mathrm{Seq}(\mathcal{K})$ and $E \in 2^\Omega$ and suppose $\mathbb{A}$ is negative, so that $A^J E \subset K^j E$ for each agent $j \in J$. Then certainly $A^J E \supset \bigcap_{j \in J} K^j E = \curlywedge^J E$. Overall, $\mathbb{A}$ is negative if and only if A^J is less informative than $\curlyvee^J$. $\square$

Proof of Lemma 2:

Fix a state space Ω and associated set of knowledge operators $\mathcal{K}$. Throughout this proof, let $(K^1, \ldots, K^n), (\tilde{K}^1, \ldots, \tilde{K}^n) \in \text{Seq}(\mathcal{K})$ such that each $\tilde{K}^i$ is more informed than each K^i. That is, suppose $K^i E \subset \tilde{K}^i E$ for all events $E \in 2^\Omega$ and $i = 1, \ldots, n$.

Consider the **somebody knows** aggregator $\curlyvee$. For any $E \in 2^\Omega$,

$$\begin{aligned} \curlyvee \left(K^1, \ldots, K^n\right) E &= \bigcup_{i=1}^{n} K^i E \\ &\subset \bigcup_{i=1}^{n} \tilde{K}^i E \\ &= \curlyvee \left(\tilde{K}^1, \ldots, \tilde{K}^n\right) E \end{aligned}$$

As this hold for all $E \in 2^\Omega$, the somebody knows aggregator is increasing.

Next, consider the **everybody knows** aggregator $\curlywedge$. For any $E \in 2^\Omega$,

$$\begin{aligned} \curlywedge \left(K^1, \ldots, K^n\right) E &= \bigcap_{i=1}^{n} K^i E \\ &\subset \bigcap_{i=1}^{n} \tilde{K}^i E \\ &= \curlywedge \left(\tilde{K}^1, \ldots, \tilde{K}^n\right) E \end{aligned}$$

As this hold for all $E \in 2^\Omega$, the everybody knows aggregator is increasing.

Next, consider the **distributed knowledge** aggregator $\vee$. For any $E \in 2^\Omega$,

$$\begin{aligned} \vee \left(K^1, \ldots, K^n\right) E &= \bigcap_{\bar{\omega} \notin E} \bigcup_{F \subset \neg\{\bar{\omega}\}} \bigcup_{i=1}^{n} K^j F \\ &\subset \bigcap_{\bar{\omega} \notin E} \bigcup_{F \subset \neg\{\bar{\omega}\}} \bigcup_{i=1}^{n} \tilde{K}^j F \\ &= \vee \left(\tilde{K}^1, \ldots, \tilde{K}^n\right) E \end{aligned}$$

As this hold for all $E \in 2^\Omega$, the distributed aggregator is increasing. $\square$

Proof of Proposition 5:

Fix a list of knowledge operators $(K^1, \ldots, K^n) \in \text{Seq}(\mathcal{K})$ and sublist $(K^1, \ldots, K^m) \in \text{Seq}(\mathcal{K})$. Let $\mathbb{A} : \text{Seq}(\mathcal{K}) \to \mathcal{K}$ be naturally positive and increasing, and fix $E \in 2^\Omega$. As $\mathbb{A}$ is naturally positive,

$$\mathbb{A}(K^1, \ldots, K^m)E = \mathbb{A}(K^1, \ldots, K^m, K^{\emptyset})E$$

Since $\mathbb{A}$ is increasing, and K^{m+1} is more informed than $K^{\emptyset}$, then

$$\mathbb{A}(K^1, \ldots, K^m, K^{\emptyset})E \subset \mathbb{A}(K^1, \ldots, K^m, K^{m+1})E$$

Continuing in this manner for $|n - m|$ steps gives

$$\mathbb{A}(K^1, \ldots, K^m)E \subset \mathbb{A}(K^1, \ldots, K^n)E$$

As this hold for all $E \in 2^{\Omega}$, $\mathbb{A}(K^1, \ldots, K^n)$ is more informative than $\mathbb{A}(K^1, \ldots, K^m)$, as required.

Similarly, let $\mathbb{A} : \mathrm{Seq}(\mathcal{K}) \to \mathcal{K}$ be naturally negative and increasing, and fix $E \in 2^{\Omega}$. As $\mathbb{A}$ is naturally negative,

$$\mathbb{A}(K^1, \ldots, K^m)E = \mathbb{A}(K^1, \ldots, K^m, K^{\Omega})E$$

Since $\mathbb{A}$ is increasing, and K^{m+1} is less informed than K^{Ω}, then

$$\mathbb{A}(K^1, \ldots, K^m, K^{\Omega})E \supset \mathbb{A}(K^1, \ldots, K^m, K^{m+1})E$$

Continuing in this manner for $|n - m|$ steps gives

$$\mathbb{A}(K^1, \ldots, K^m)E \supset \mathbb{A}(K^1, \ldots, K^n)E$$

As this hold for all $E \in 2^{\Omega}$, $\mathbb{A}(K^1, \ldots, K^m)$ is more informative than $\mathbb{A}(K^1, \ldots, K^n)$, as required. $\square$

Proof of Proposition 6:

Let $\mathcal{L} \subset \mathcal{K}$, and suppose aggregator $\mathbb{A} : \mathrm{Seq}(\mathcal{K}) \to \mathcal{K}$ is a 1-identity on $\mathcal{L}$, and stationary. Let $L \in \mathcal{L}$. As $\mathbb{A}$ is a 1-identity on $\mathcal{L}$, then $\mathbb{A}(L) = L$. Since operator L is weakly less informed than L, and also weakly more informed than L, and as $\mathbb{A}$ is stationary, therefore $\mathbb{A}(L, L) = \mathbb{A}(L) = L$. Continuing by induction we have

$$\mathbb{A}(\underbrace{L, L, \ldots, L}_{k \text{ times}}) = L$$

for all $k \in \mathbb{N}$. So $\mathbb{A}$ is an identity on $\mathcal{L}$, as required. $\square$

Proof of Lemma 3:

Fix a correspondence operator $K \in \mathcal{K}_C$. There exists a correspondence $\gamma : \Omega \to 2^\Omega$ such that $KE = \{\omega \in \Omega \mid \gamma(\omega) \subset E\}$ for all events $E \in 2^\Omega$. Then

$$K\Omega = \{\omega \in \Omega \mid \gamma(\omega) \subset \Omega\} = \Omega$$

For every pair of events $E, F \in 2^\Omega$,

$$\begin{aligned} K(E \cap F) &= \{\omega \in \Omega \mid \gamma(\omega) \subset E \cap F\} \\ &= \{\omega \in \Omega \mid \gamma(\omega) \subset E\} \cap \{\omega \in \Omega \mid \gamma(\omega) \subset F\} \\ &= KE \cap KF \end{aligned}$$

Therefore $K\Omega = \Omega$ and $K(E \cap F) = KE \cap KF)$, as required.

Now fix an operator $K \in \mathcal{K}$ such that $K\Omega = \Omega$ and $K(E \cap F) = KE \cap KF$ for all events $E, F \in 2^\Omega$. Let $\mathcal{E} : \Omega \to 2^{2^\Omega}$ be given by Equation 8, so that for $\omega \in \Omega$,

$$\mathcal{E}(\omega) = \{E \in 2^\Omega \mid \omega \in KE\}$$

As $K\Omega = \Omega$, then $\Omega \in \mathcal{E}(\omega)$ for all $\omega \in \Omega$. In particular, $\mathcal{E}(\omega)$ is non-empty. Define the correspondence $\gamma : \Omega \to 2^\Omega$ by

$$\gamma(\omega) = \bigcap_{F \in \mathcal{E}(\omega)} F$$

Let K_γ be given by Equation 6, so that for all events $E \in 2^\Omega$,

$$K_\gamma E = \{\omega \in \Omega \mid \gamma(\omega) \subset E\}$$

First we show $KE \subset K_\gamma E$. Let $\omega \in KE$. By definition, $\mathcal{E} \in \mathcal{E}(\omega)$. As $E \in \mathcal{E}(\omega)$, then certainly the intersection over all events in $\mathcal{E}(\omega)$ is a subset of E. That is

$$\bigcap_{F \in \mathcal{E}(\omega)} \subset E$$

As $\bigcap_{F \in \mathcal{E}(\omega)} = \gamma(\omega)$ we have $\gamma(\omega) \subset E$. Therefore $\omega \in K_\gamma E$.

Next we show $KE \supset K_\gamma E$. Let $\omega \in K_\gamma E$. By definition, $\gamma(\omega) \subset E$. As $K(F_1 \cap F_2) = KF_1 \cap KF_2$ for all events $F_1, F_2 \in 2^\Omega$, for each $\omega \in \Omega$, the set $\mathcal{E}(\omega)$ is closed under intersections, and supersets. That is, $F_1, F_2 \in \mathcal{E}(\omega)$ implies $F_1 \cap F_2 \in \mathcal{E}(\omega)$; and $F_1 \in \mathcal{E}(\omega)$ with $F_1 \subset F_2$ implies $F_2 \in \mathcal{E}(\omega)$. As $\gamma(\omega)$ is the intersection of events in $\mathcal{E}(\omega)$, therefore $\gamma(\omega) \in \mathcal{E}(\omega)$. As $\gamma(\omega) \subset E$, therefore $E \in \mathcal{E}(\omega)$. That is, $\omega \in KE$, as required. $\square$

Proof of Lemma 5:

The distributed knowledge aggregator is given by Equation 1 as

$$\vee(K^1, \dots, K^n)E = \bigcap_{\bar{\omega} \notin E} \bigcup_{F \subset \neg\{\bar{\omega}\}} \bigcup_{i=1}^{n} K^i F$$

Suppose K^i satisfies Monotonicity for all $i = 1, \dots, n$. Fix $\bar{\omega} \in \Omega$. For all $F \subset \neg\{\bar{\omega}\}$, then $K^i F \subset K^i \neg\{\bar{\omega}\}$ as K^i is Monotonic. As $\neg\{\bar{\omega}\} \subset \neg\{\bar{\omega}\}$, for each $i \in 1, \dots, n$,

$$\bigcup_{F \subset \neg\{\bar{\omega}\}} K^i F = K^i \neg\{\bar{\omega}\}$$

Therefore

$$\bigcap_{\bar{\omega} \notin E} \bigcup_{F \subset \neg\{\bar{\omega}\}} \bigcup_{i=1}^{n} K^i F = \bigcap_{\bar{\omega} \notin E} \bigcup_{j \in J} K^j \neg\{\bar{\omega}\}$$

as required. $\square$

Proof of Proposition 9:

This result is presented as Theorem 1 of Bacharach [3]. The proof is not reproduced here.

Proof of Lemma 4:

Let $\gamma : \Omega \to 2^\Omega$ satisfy axioms P.1 and P.2, and let K_γ be the knowledge operator defined by Equation 6. From Lemma 3, K satisfies axioms K.0, K.1, and K.2. By Bacharach's Theorem it remains to show that K_γ satisfies axioms K.3, K.4, and K.5.

For axiom K.3: Let $\omega \in K_\gamma E$ so that $\gamma(\omega) \subset E$. By P.1, $\omega \in \gamma(\omega)$, and thus $\omega \in E$. Therefore, $K_\gamma E \subset E$, and $K_\gamma \in \mathcal{K}_3$.

Before we proceed to $\mathcal{K}_4$, notice that if $\gamma(\omega) = \gamma(\omega')$, then $\omega \in K_\gamma E$ if and only if $\omega' \in K_\gamma E$. Conversely, if $\gamma(\omega) \cap \gamma(\omega') = \emptyset$, and given $\omega \in \gamma(\omega)$, then $\omega' \notin \gamma(\omega)$. Therefore, if $\omega' \in \gamma(\omega)$, then $\omega \in K_\gamma E$ if and only if $\omega' \in K_\gamma E$.

For axiom K.4: Let $\omega \notin K_\gamma K_\gamma E$. Then, $\gamma(\omega) \not\subset K_\gamma E$, and so, there exists $\omega' \in \gamma(\omega)$ such that $\omega' \notin KE$. From the preceding note, $\omega \notin K_\gamma E$, and so, $\neg K_\gamma K_\gamma E \subset \neg K_\gamma E$, as required.

For axiom K.5: Let $\omega \notin K_\gamma \neg K_\gamma E$. Then, $\gamma(\omega) \not\subset \neg K_\gamma E$. There is a state $\omega' \in \gamma(\omega)$ such that $\omega' \notin \neg K_\gamma E$, which is $\omega' \in K_\gamma E$. From the preceding note, $\omega \in K_\gamma E$, and so, $\neg K_\gamma \neg K_\gamma E \subset K_\gamma E$, as required. □

Proof of Proposition 10:

Somebody knows and Everybody knows are clearly identities on $\mathcal{K}$ as they are just the union and intersection, respectively, of a single knowledge operator.

Now we want to show **distributed knowledge**, $\vee$, is an identity on $\mathcal{K}_C$. As $\vee$ is stationary, by Proposition 6, it is enough to show $\vee$ is a 1-identity on $\mathcal{K}_C$. Let $K \in \mathcal{K}_C$. Then

$$\vee(K)E = \bigcap_{\bar{\omega} \notin E} \bigcup_{F \subset \neg\{\bar{\omega}\}} KE$$

As $K \in \mathcal{K}_C$ implies K satisfies Monotonicity, then by Lemma 5, the aggregator $\vee$ can be written in the form given by Equation 7.

$$\vee(K)E = \bigcap_{\bar{\omega} \notin E} K(\neg\{\bar{\omega}\})$$

$K \in \mathcal{K}_C$ also implies K satisfies Conjunction. Therefore

$$\begin{aligned} \vee(K)E &\subset K\left(\bigcap_{\bar{\omega} \notin E} \neg\{\bar{\omega}\}\right) \\ &= KE \end{aligned}$$

From Lemma 1, $KE \subset \vee(K)E$. Therefore, $\vee(K) = K$ and $\vee$ is a 1-identity, and thus an identity, on $\mathcal{K}_C$.

Similarly, we want to show **common knowledge**, $\wedge$, is an identity on $\mathcal{K}_4$. As $\wedge$ is stationary, by Proposition 6, it is enough to show $\wedge$ is a 1-identity on $\mathcal{K}_4$. Let $K \in \mathcal{K}_4$, so that $KE \subset KKE$ for all events $E \in 2^\Omega$. As $\curlywedge(K) = K$, we have $(\curlywedge(K))^{(s)} = K^{(s)}$ for all $s \in \mathbb{N}$. Therefore, for all events $E \in 2^\Omega$,

$$\begin{aligned}
\wedge(K)E &= \bigcap_{s=1}^{\infty}(\curlywedge(K))^{(s)}E \\
&= \bigcap_{s=1}^{\infty} K^{(s)}E \\
&= KE \cap KKE \cap KKKE \cap \ldots \\
&= KE
\end{aligned}$$

Therefore $\wedge(K) = K$ and $\wedge$ is a 1-identity, and thus an identity, on $\mathcal{K}_4$. □

Proof of Lemma 6:

Let $(K^1, \ldots, K^n) \in \text{Seq}(\mathcal{K}_1 \cap \mathcal{K}_4)$. We want to show that the somebody knows aggregator $\curlyvee$ satisfies Positive Introspection.

Fix an event $E \in 2^\Omega$. For each $j \in 1, \ldots, n$, operator K^j satisfies Positive Introspection, so $K^jE \subset K^jK^jE$. Since K^j is Monotonic,

$$K^jE \subset K^jK^jE \subset K^j \left[\bigcup_{i=1}^{n} K^iE\right]$$

Taking the union of both sides over all $j = 1, \ldots, n$ yields

$$\bigcup_{j=1}^{n} K^jE \subset \bigcup_{j=1}^{n} \left(K^j \left[\bigcup_{i=1}^{n} K^iE\right]\right)$$

which is to say

$$\curlyvee\left(K^1, \ldots, K^n\right)E \subset \curlyvee\left(K^1, \ldots, K^n\right)\left[\curlyvee\left(K^1, \ldots, K^n\right)\right]E$$

As this hold for all $E \in 2^\Omega$, the somebody knows aggregator preserves Positive Introspection if all input operators are Monotonic. □

Proof of Proposition 11:

Throughout, let $(\Omega, J, \{K^j\}_{j\in J}$ be a knowledge model.

First, consider the **everybody knows** aggregation, $\curlywedge^J E = \bigcap_{j\in J} K^j E$, for every $E \in 2^\Omega$. We want to show the everybody knows aggregator preserves axioms K.0, K.1, K.2, and K.3, and does not preserve K.4 or K.5.

K.0: Suppose $K^j\Omega = \Omega$ for all $j \in J$. Then

$$\curlywedge^J \Omega = \bigcap_{j\in J} K^j\Omega = \bigcap_{j\in J} \Omega = \Omega$$

So $\curlywedge^J$ satisfies Awareness, and thus aggregator $\curlywedge$ preserves axiom K.0.

K.1: Suppose K^j satisfies Monotonicity for all $j \in J$. Let $E \subset F$, so that $K^jE \subset K^jF$ for all $j \in J$. Then

$$\curlywedge^J E = \bigcap_{j\in J} K^jE \subset \bigcap_{j\in J} K^jF = \curlywedge^J F$$

So $\curlywedge^J$ satisfies Monotonicity, and thus aggregator $\curlywedge$ preserves axiom K.1.

K.2: Suppose K^j satisfies Conjunction for all $j \in J$, so that $K^jE \cap K^jF \subset K^j(E\cap F)$ for all $E, F \in 2^\Omega$. Then

$$\begin{aligned}
\curlywedge^J E \cap \curlywedge^J F &= \bigcap_{j\in J} K^jE \cap \bigcap_{j\in J} K^jF \\
&= \bigcap_{j\in J} (K^jE \cap K^jF) \\
&\subset \bigcap_{j\in J} K^j(E\cap F) \\
&= \curlywedge^J (E\cap F)
\end{aligned}$$

So $\curlywedge^J$ satisfies Conjunction, and thus aggregator $\curlywedge$ preserves axiom K.2.

K.3: Suppose K^j satisfies Truth for all $j \in J$, so that $K^jE \subset E$ for all $E \in 2^\Omega$. Then

$$\curlywedge^J E = \bigcap_{j\in J} K^jE \subset \bigcap_{j\in J} E = E$$

So $\curlywedge^J$ satisfies Truth, and thus aggregator $\curlywedge$ preserves axiom K.3.

K.4: In Example 6, each agents' knowledge operator satisfies Positive Introspection, but $\curlywedge^J$ does not. Therefore, $\curlywedge$ does not preserve axiom K.4.

K.5: We construct an example where each agent's knowledge operator satisfies Negative Introspection, but the aggregation $\curlywedge^J$ does not. Let $\Omega = \{a, b, c\}$, $J = \{1, 2\}$, and $K^1, K^2 : 2^\Omega \to 2^\Omega$ be

$$\begin{aligned} K^1\emptyset &= \{a, c\}, & K^2\emptyset &= \{b, c\} & \\ K^1\{a, b\} &= \emptyset, & K^2\{a, b\} &= \emptyset & \\ K^1\{c\} &= \{a, b, c\}, & K^2\{c\} &= \{a, b, c\} & \\ K^1E &= E, & K^2E &= E, & \text{otherwise} \end{aligned}$$

By inspection, the knowledge operators K^1 and K^2 satisfy Negative Introspection; in each case $\neg K^j E \subset K^j \neg K^j E$ for all $E \in 2^\Omega$. However

$$\neg \curlywedge^J \emptyset = \neg \left(K^1\emptyset \cap K^2\emptyset\right) = \neg \left(\{a, c\} \cap \{b, c\}\right) = \neg\{c\} = \{a, b\}$$

While

$$\curlywedge^J \neg \curlywedge^J \emptyset = K^1\{a, b\} \cap K^2\{a, b\} = \emptyset \cap \emptyset = \emptyset$$

Thus $\neg \curlywedge^J \emptyset \not\subset \curlywedge^J \neg \curlywedge^J \emptyset$, so $\curlywedge^J$ does not satisfy Negative Introspection. Therefore, $\curlywedge$ does not preserve axiom K.5.

Now consider the **common knowledge** aggregation, $\wedge^J E = \bigcap_{s=1}^\infty \left(\curlywedge^J\right)^{(s)} E$, for every $E \in 2^\Omega$. We want to show the common knowledge aggregator preserves axioms K.0, K.1, and K.3, and does not preserve K.2, K.4 or K.5.

K.0: Suppose $K^j\Omega = \Omega$ for all $j \in J$. As $\curlywedge$ preserves Awareness, $\curlywedge^J\Omega = \Omega$. Then

$$\wedge^J \Omega = \bigcap_{s=1}^\infty \left(\curlywedge^J\right)^{(s)} \Omega = \bigcap_{s=1}^\infty \Omega = \Omega$$

So $\wedge^J$ satisfies Awareness, and thus aggregator $\wedge$ preserves axiom K.0.

K.1: Suppose K^j satisfies Monotonicity for all $j \in J$. Let $E \subset F$, so that $K^jE \subset K^jF$ for all $j \in J$. As $\curlywedge$ preserves Monotonicity, $\curlywedge^J E \subset \curlywedge^J F$. Then

$$E \subset F \implies \curlywedge^J E \subset \curlywedge^J F \implies \left(\curlywedge^J\right)^{(2)} E \subset \left(\curlywedge^J\right)^{(2)} F \implies \cdots$$

So $\left(\curlywedge^J\right)^{(s)} E \subset \left(\curlywedge^J\right)^{(s)} F$ for all $s \in \mathbb{N}$. Therefore

$$\wedge^J E = \bigcap_{s=1}^\infty \left(\curlywedge^J\right)^{(s)} E \subset \bigcap_{s=1}^\infty \left(\curlywedge^J\right)^{(s)} F = \wedge^J F$$

So $\wedge^J$ satisfies Monotonicity, and thus aggregator $\wedge$ preserves axiom K.1.

K.2: We construct an example where each agent's knowledge operator satisfies Conjunction, but the aggregation $\wedge^J$ does not. Let $\Omega = \{a, b, c\}$, $J = \{1\}$, and $K^1 : 2^\Omega \to 2^\Omega$ be

$$\begin{aligned} K^1\{a,b\} &= \{a,b\} \\ K^1\{a,c\} &= \{a,c\} \\ K^1\{a\} &= \{a,b,c\} \\ KE &= \emptyset, \quad \text{otherwise} \end{aligned}$$

By inspection K^1 satisfies Conjunction. In each case $K^1E \cap K^1F \subset K^1(E \cap F)$ for all events $E, F \in 2^\Omega$. For $E = \{a, b\}$ and $F = \{a, c\}$ the common knowledge aggregation $\wedge^J$ has $\wedge^J\{a,b\} \cap \wedge^J = \{a,b\} \cap \{a,c\} = \{a\}$. However $\wedge^J\{a\} \subset K\left(K\{a\}\right) = \emptyset$, so $\wedge^J\{a\} = \emptyset$. As

$$\{a\} = \wedge^J\{a,b\} \cap \wedge^J\{a,c\} \not\subset \wedge^J\left(\{a,b\} \cap \{a,c\}\right) = \emptyset$$

so $\wedge^J$ does not satisfy Conjunction. Therefore, $\wedge$ does not preserve axiom K.2.

K.3: Suppose K^j satisfies Truth for all $j \in J$, so that $K^jE \subset E$ for all $E \in 2^\Omega$. As $\curlywedge$ preserves Truth, $\curlywedge^J E \subset E$ for all events $E \in 2^\Omega$. Then

$$E \supset \curlywedge^J E \supset \left(\curlywedge^J\right)^{(2)} E \supset \cdots$$

So $\left(\curlywedge^J\right)^{(s)} E \subset E$ for all $s \in \mathbb{N}$. Therefore

$$\wedge^J E = \bigcap_{s=1}^{\infty} \left(\curlywedge^J\right)^{(s)} E \subset E$$

So $\wedge^J$ satisfies Truth, and thus aggregator $\wedge$ preserves axiom K.3.

K.4: We construct an example where each agent's knowledge operator satisfies Positive Introspection, but the aggregation $\wedge^J$ does not. Let $\Omega = \{a, b, c, d, e\}$, $J = \{1, 2\}$, and $K^1, K^2 : 2^\Omega \to 2^\Omega$ be

$$\begin{aligned} K^1\{a,b\} &= \{a,c,d\}, & K^2\{a,b\} &= \{a,c,e\} \\ K^1\{a,c\} &= \{a,b,d\}, & K^2\{a,c\} &= \{a,b,e\} \\ K^1\{a\} &= \emptyset, & K^2\{a\} &= \emptyset \\ K^1E &= E, & K^2E &= E, & &\text{otherwise} \end{aligned}$$

By inspection, both K^1 and K^2 satisfy Positive Introspection. The everybody knows aggregation $\curlywedge^J$ has

$$\curlywedge^J\{a,b\} = \{a,c\},\ \curlywedge^J\{a,c\} = \{a,b\},\ \curlywedge^J\{a\} = \emptyset,\ \text{and}\ \curlywedge^J\emptyset = \emptyset$$

From this, the common knowledge aggregation $\wedge^J$ has

$$\wedge^J\{a,b\} = \{a\},\ \wedge^J\{a\} = \emptyset$$

As $\wedge^J\{a,b\} \not\subset \wedge^J \wedge^J \{a,b\}$, the aggregation $\wedge^J$ does not satisfy Positive Introspection. Therefore, $\wedge$ does not preserve axiom K.4.

K.5: We construct an example where each agent's knowledge operator satisfies Negative Introspection, but the aggregation $\wedge^J$ does not. Let $\Omega = \{a,b,c\}$, $J = \{1,2\}$, and $K^1, K^2 : 2^\Omega \to 2^\Omega$ be

$$\begin{array}{rrl} K^1\emptyset = \{a,c\}, & K^2\emptyset = \{b,c\} & \\ K^1\{a,b\} = \emptyset, & K^2\{a,b\} = \emptyset & \\ K^1\{c\} = \{a,b,c\}, & K^2\{c\} = \{a,b,c\} & \\ K^1E = E, & K^2E = E, & \quad \text{otherwise} \end{array}$$

By inspection, the knowledge operators K^1 and K^2 satisfy Negative Introspection; in each case $\neg K^jE \subset K^j\neg K^jE$ for all $E \in 2^\Omega$. However

$$\begin{aligned} \wedge^J\emptyset &= \curlywedge^J\emptyset \cap \curlywedge^J \curlywedge^J \emptyset \cap \cdots \\ &= \{c\} \cap \curlywedge^J\{c\} \cap \curlywedge^J \curlywedge^J \{c\} \cap \cdots \\ &= \{c\} \cap \{a,b,c\} \cap \{a,b,c\} \cap \cdots = \{c\} \end{aligned}$$

so $\neg \wedge^J \emptyset = \{a,b\}$. While

$$\begin{aligned} \wedge^J\{a,b\} &= \curlywedge^J\{a,b\} \cap \curlywedge^J \curlywedge^J \{a,b\} \cap \cdots \\ &= \emptyset \cap \curlywedge^J\emptyset \cap \curlywedge^J \curlywedge^J \emptyset \cap \cdots \\ &= \emptyset \cap \emptyset \cap \emptyset \cap \cdots = \emptyset \end{aligned}$$

so $\wedge^J\neg \wedge^J \emptyset = \emptyset$. Thus $\neg \wedge^J \emptyset \not\subset \wedge^J\neg \wedge^J \emptyset$, and $\wedge^J$ does not satisfy Negative Introspection. Therefore, $\wedge$ does not preserve axiom K.5.

Now consider the **somebody knows** aggregation, $\curlyvee^J E = \bigcup_{j\in J} K^j E$, for every $E \in 2^\Omega$. We want to show the somebody knows aggregator preserves axioms K.0, K.1, and K.3, and does not preserve K.2, K.4 or K.5.

K.0: Suppose $K^j\Omega = \Omega$ for all $j \in J$. Then

$$\curlyvee^J \Omega = \bigcup_{j\in J} K^j\Omega = \bigcup_{j\in J} \Omega = \Omega$$

So $\curlyvee^J$ satisfies Awareness, and thus aggregator $\curlyvee$ preserves axiom K.0.

K.1: Suppose K^j satisfies Monotonicity for all $j \in J$. Let $E \subset F$, so that $K^jE \subset K^jF$ for all $j \in J$. Then

$$\curlyvee^J E = \bigcup_{j\in J} K^jE \subset \bigcup_{j\in J} K^jF = \curlyvee^J F$$

So $\curlyvee^J$ satisfies Monotonicity, and thus aggregator $\curlyvee$ preserves axiom K.1.

K.2: In Example 6, each agents' knowledge operator satisfies Conjunction, but $\curlyvee^J$ does not. Therefore,$\curlyvee$ does not preserve axiom K.4.

K.3: Suppose K^j satisfies Truth for all $j \in J$, so that $K^jE \subset E$ for all $E \in 2^\Omega$. Then

$$\curlyvee^J E = \bigcup_{j\in J} K^jE \subset \bigcup_{j\in J} E = E$$

So $\curlyvee^J$ satisfies Truth, and thus aggregator $\curlyvee$ preserves axiom K.3.

K.4: We construct an example where each agent's knowledge operator satisfies Positive Introspection, but the aggregation $\curlyvee^J$ does not. Let $\Omega = \{a, b\}$, $J = \{1, 2\}$, and $K^1, K^2 : 2^\Omega \to 2^\Omega$ be

$$\begin{aligned} K^1\emptyset &= \emptyset, & K^2\emptyset &= \emptyset \\ K^1\{a\} &= \{a\}, & K^2\{a\} &= \{b\} \\ K^1\{b\} &= \{b\}, & K^2\{b\} &= \{b\} \\ K^1\{a,b\} &= \emptyset, & K^2\{a,b\} &= \emptyset \end{aligned}$$

By inspection, the knowledge operators K^1 and K^2 satisfy Positive Introspection; in each case $K^jE \subset K^jK^jE$ for all $E \in 2^\Omega$. However

$$\curlyvee^J\{a\} = K^1\{a\} \cup K^2\{a\} = \{a\} \cup \{b\} = \{a, b\}$$

While

$$\curlyvee^J \curlyvee^J \{a\} = K^1\{a,b\} \cup K^2\{a,b\} = \emptyset \cup \emptyset = \emptyset$$

Thus $\curlyvee^J\{a\} \not\subset \curlyvee^J \curlyvee^J \{a\}$, so $\curlyvee^J$ does not satisfy Positive Introspection. Therefore, $\curlyvee$ does not preserve axiom K.4.

K.5: We construct an example where each agent's knowledge operator satisfies Negative Introspection, but the aggregation $\curlyvee^J$ does not. Let $(\Omega, J, \{K^j\}_{j\in J})$ be as in Example 6. As the knowledge operators K^1, K^2 are partitional, by Proposition 9 they satisfy Negative Introspection. However, $\neg\curlyvee\{a,c\} = \{b\}$ while $\curlyvee^J\neg\curlyvee\{a,c\} = \emptyset$. Thus $\neg\curlyvee^J\{a,c\} \not\subset \curlyvee^J\neg\curlyvee^J\{a,c\}$, so $\curlyvee^J$ does not satisfy Negative Introspection. Therefore, $\curlyvee$ does not preserve axiom K.5.

Finally, consider the **distributed knowledge** aggregation, $\vee^J E = \bigcap_{\bar\omega\notin E} \bigcup_{F\subset\neg\{\omega\}} \curlyvee^J F$, for every $E \in 2^\Omega$. We want to show the distributed knowledge aggregator forces axioms K.0, K.1, and K.2, and preserves K.3, K.4 and K.5.

K.0: $\vee^J\Omega$ is an empty intersection, and so, $\vee^J\Omega = \Omega$ regardless of any assumptions of each K^j. Therefore the distributed knowledge aggregator $\vee$ forces axiom K.0.

K.1: Let $E_1, E_2 \in 2^\Omega$, such that $E_1 \subset E_2$. As each state $\bar\omega$ not in E_1 is either outside E_2, or inside E_2 but outside E_1, then

$$\begin{aligned}
\vee^J E_1 &= \bigcap_{\bar\omega\notin E_1} \bigcup_{F\subset\neg\{\omega\}} \curlyvee^J F \\
&= \left(\bigcap_{\bar\omega\notin E_2} \bigcup_{F\subset\neg\{\omega\}} \curlyvee^J F\right) \cap \left(\bigcap_{\bar\omega\in(E_2\cap\neg E_1)} \bigcup_{F\subset\neg\{\omega\}} \curlyvee^J F\right) \\
&\subset \bigcap_{\bar\omega\notin E_2} \bigcup_{F\subset\neg\{\omega\}} \curlyvee^J F \\
&= \vee^J E_2
\end{aligned}$$

So $\vee^J$ satisfies Monotonicity regardless of any assumptions on each K^j. Therefore the distributed knowledge aggregator $\vee$ forces axiom K.1.

K.2: Let $E_1, E_2 \subset \Omega$. Then

$$\begin{aligned}
\vee^J E_1 \cap \vee^J E_2 &= \left(\bigcap_{\bar{\omega} \notin E_1} \bigcup_{F \subset \neg\{\omega\}} \curlyvee^J F \right) \cap \left(\bigcap_{\bar{\sigma} \notin E_2} \bigcup_{F \subset \neg\{\bar{\sigma}\}} \curlyvee^J F \right) \\
&= \bigcap_{\bar{\omega} \notin E_1 \cap E_2} \bigcup_{F \subset \neg\{\bar{\omega}\}} \curlyvee^J F \\
&= \vee^J (E_1 \cap E_2)
\end{aligned}$$

So $\vee^J$ satisfies Conjunction regardless of any assumptions on each K^j. Therefore the distributed knowledge aggregator $\vee$ forces axiom K.2.

K.3: Suppose K^j satisfies Truth for all $j \in J$, so that $K^j E \subset E$ for all $E \in 2^\Omega$. As $\curlyvee$ preserves Truth, $\curlyvee E \subset E$ for all events $E \in 2^\Omega$. Then

$$\begin{aligned}
\vee^J E &= \bigcap_{\bar{\omega} \notin E} \bigcup_{F \subset \neg\{\omega\}} \curlyvee^J F \\
&\subset \bigcap_{\bar{\omega} \notin E} \bigcup_{F \subset \neg\{\omega\}} F
\end{aligned}$$

As the union of all subsets of $\neg\{\bar{\omega}\}$ is just $\neg\{\bar{\omega}\}$,

$$\begin{aligned}
&= \bigcap_{\bar{\omega} \notin E} \neg\{\bar{\omega}\} \\
&= E
\end{aligned}$$

So $\vee^J$ satisfies Truth, and thus aggregator $\vee$ preserves axiom K.3.

For properties K.4 and K.5, by Proposition 2, the distributed knowledge aggregator $\vee$ is associative. In particular,

$$\vee \left(K^1, \cdots, K^{|J|} \right) = \vee \left(\vee(K^1), \cdots, \vee(K^{|J|}) \right)$$

As $\vee$ forces Monotonicity, so $\vee(K^j)$ is monotonic, we may assume, without loss of generality, that operator K^j is monotonic. Further, when each K^j is monotonic then by Lemma 5, for all events $E \in 2^\Omega$,

$$\vee^J E = \bigcap_{\bar{\omega} \notin E} \curlyvee^J (\neg\{\bar{\omega}\})$$

K.4: Suppose K^j satisfies Positive Introspection for all $j \in J$, so that $K^j E \subset K^j K^j E$ for all $E \in 2^\Omega$. Assume, without loss of generality, that K^j satisfies Monotonicity for each $j \in J$. By Lemma 6, $\curlyvee^J$ then satisfies Positive Introspection, so $\curlyvee^J E \subset \curlyvee^J \curlyvee^J E$ for all events $E \in 2^\Omega$. By Lemma 1, $\curlyvee^J \curlyvee^J E \subset \vee^J \curlyvee^J E$, so $\curlyvee^J E \subset \vee^J \curlyvee^J E$ for all events $E \in 2^\Omega$. Then

$$\begin{aligned} \vee^J E &= \bigcap_{\omega \notin E} \curlyvee^J(\neg\{\omega\}) \\ &\subset \bigcap_{\omega \notin E} \vee^J \curlyvee^J (\neg\{\omega\}) \end{aligned}$$

As $\vee$ forces K.1 and K.2, the aggregation $\vee^J$ distributes over intersections. Therefore

$$\begin{aligned} &= \vee^J \bigcap_{\omega \notin E} \curlyvee^J(\neg\{\omega\}) \\ &= \vee^J \vee^J E \end{aligned}$$

So $\vee^J$ satisfies Positive Introspection, and thus aggregator $\vee$ preserves axiom K.4.

K.5: Suppose K^j satisfies Negative Introspection for all $j \in J$, so that $\neg K^j E \subset K^j \neg K^j E$ for all $E \in 2^\Omega$. Assume, without loss of generality, that K^j satisfies Monotonicity for each $j \in J$. Using de Morgan's Laws we have

$$\begin{aligned} \neg \vee^J E &= \neg \left(\bigcap_{\omega \notin E} \bigcup_{j \in J} (\neg\{\omega\})) \right) \\ &= \bigcup_{\omega \notin E} \bigcap_{j \in J} \neg K^j(\neg\{\omega\}) \end{aligned}$$

As each $\neg K^j E \subset K^j \neg K^j E$ for all $j \in J$, and as $\vee^J$ is a positive aggregator

$$\begin{aligned} &\subset \bigcup_{\omega \notin E} \bigcap_{j \in J} K^j \neg K^j(\neg\{\omega\}) \\ &\subset \bigcup_{\omega \notin E} \bigcap_{j \in J} \vee^J \neg K^j(\neg\{\omega\}) \end{aligned}$$

Since $\vee$ forces K.1 and K.2, the aggregation $\vee^J$ distributes over intersections. Then

$$= \bigcup_{\omega \notin E} \vee^J \bigcap_{j \in J} \neg K^j(\neg\{\omega\})$$

As $\vee^J$ satisfies Monotonicity, for any sets $E_1, \ldots, E_n$, then $\vee^J E_1 \cup \ldots \cup \vee^J E_n \subset \vee^J(E_1 \cup \ldots \cup E_n)$. In particular

$$\subset \vee^J \left(\bigcup_{\omega \notin E} \bigcap_{j \in J} \neg K^j(\neg\{\omega\}) \right)$$

and again using de Morgan's laws gives

$$\begin{aligned} &= \vee^J \left(\neg \bigcap_{\omega \notin E} \curlyvee^J(\neg\{\omega\}) \right) \\ &= \vee^J \neg \vee^J E \end{aligned}$$

So $\vee^J$ satisfies Negative Introspection, and thus aggregator $\vee$ preserves axiom K.5. $\square$

Proof of Proposition 8:

Fix a state space Ω and let $\mathcal{K}$ be the associated set of knowledge operators. Let $(K^1, \ldots, K^n) \in \text{Seq}(\mathcal{K}_C)$, so that all operators in the sequence are correspondence operators. By Lemma 3 each K^i is in $\mathcal{K}_0 \cap \mathcal{K}_1 \cap K_2$.

Consider the **distributed knowledge** aggregator $\vee$. As $(K^1, \ldots, K^n) \in \text{Seq}(\mathcal{K}_l)$ for $l = 0, 1, 2$, and as $\vee$ is, by Proposition 11, $\mathcal{K}_l$-preserving for $l = 0, 1, 2$, therefore $\vee(K^1, \ldots, K^n) \in \mathcal{K}_l$ for $l = 0, 1, 2$. Therefore by Lemma 3 $\vee(K^1, \ldots, K^n) \in \mathcal{K}_C$. As this is true for all sequences $(K^1, \ldots, K^n) \in \text{Seq}(\mathcal{K}_C)$, the distributed knowledge aggregator is correspondence-preserving.

Similarly by Proposition 11 and Lemma 3, the **everybody knows** aggregator $\curlywedge$ is correspondence-preserving.

Consider the **common knowledge** aggregator $\wedge$. As $(K^1, \ldots, K^n) \in \text{Seq}(\mathcal{K}_l)$ for $l = 0, 1$, and as $\wedge$ is, by Proposition 11, $\mathcal{K}_l$-preserving for $l = 0, 1$, therefore $\vee(K^1, \ldots, K^n) \in \mathcal{K}_l$ for $l = 0, 1$. Now we turn to axiom K.2, the Conjunction axiom. By Proposition 11, the somebody knows aggregator preserves axioms K.1 and K.2, so

$$\curlywedge(K^1, \ldots, K^n)(E \cap F) = \curlywedge(K^1, \ldots, K^n)E \cap \curlywedge(K^1, \ldots, K^n)F$$

Suppose, for the purposes of induction, that $(\curlywedge(K^1, \ldots, K^n))^{(s)}$ preserves axioms K.1 and

K.2 for some $s \in \mathbb{N}$. Then

$$
\begin{aligned}
&\left(\curlywedge(K^1, \dots, K^n)\right)^{(s+1)} (E \cap F) \\
&= \curlywedge (K^1, \dots, K^n) \left[\left(\curlywedge(K^1, \dots, K^n)\right)^{(s)} (E \cap F)\right] \\
&= \curlywedge (K^1, \dots, K^n) \left[\left(\curlywedge(K^1, \dots, K^n)\right)^{(s)} E \cap \left(\curlywedge(K^1, \dots, K^n)\right)^{(s)} F)\right] \\
&= \curlywedge (K^1, \dots, K^n) \left[\left(\curlywedge(K^1, \dots, K^n)\right)^{(s)} E\right] \cap \curlywedge(K^1, \dots, K^n) \left[\left(\curlywedge(K^1, \dots, K^n)\right)^{(s)} F\right] \\
&= \left(\curlywedge(K^1, \dots, K^n)\right)^{(s+1)} E \cap \left(\curlywedge(K^1, \dots, K^n)\right)^{(s+1)} F
\end{aligned}
$$

By induction over s, $(\curlywedge(K^1, \dots, K^n))^{(s)}$ satisfies axiom K.1 and K.2 for all $s \in \mathbb{N}$. Therefore

$$
\begin{aligned}
\wedge(K^1, \dots, K^n)(E \cap F) &= \bigcap_{s=1}^{\infty} \left(\curlywedge(K^1, \dots, K^n)\right)^{(s)} (E \cap F) \\
&= \bigcap_{s=1}^{\infty} \left(\curlywedge(K^1, \dots, K^n)\right)^{(s)} E \cap \bigcap_{s=1}^{\infty} \left(\curlywedge(K^1, \dots, K^n)\right)^{(s)} F \\
&= \wedge(K^1, \dots, K^n) E \cap \wedge(K^1, \dots, K^n) F
\end{aligned}
$$

so $\wedge$ satisfies axiom K.1 and K.2. Therefore by Lemma 3 $\wedge(K^1, \dots, K^n) \in \mathcal{K}_C$. As this is true for all sequences $(K^1, \dots, K^n) \in \text{Seq}(\mathcal{K}_C)$, the common knowledge aggregator is correspondence-preserving. □

Proof of Proposition 7:

Fix a state space Ω and let $\mathcal{K}$ be the associated set of knowledge operators. Let $(K^1, \dots, K^n) \in \text{Seq}(\mathcal{K}_P)$, so that all operators in the sequence are partitional. By Proposition 9 each K^i is in $\mathcal{K}_0 \cap \mathcal{K}_1 \cap K_2 \cap \mathcal{K}_3 \cap \mathcal{K}_4 \cap \mathcal{K}_5$.

Consider the **distributed knowledge** aggregator $\vee$. As $(K^1, \dots, K^n) \in \text{Seq}(\mathcal{K}_l)$ for $l = 0, \dots 5$, and as $\vee$ is, by Proposition 11, $\mathcal{K}_l$-preserving for $l = 0, \dots, 5$. Consequently, $\vee(K^1, \dots, K^n) \in \mathcal{K}_l$ for $l = 0, \dots, 5$. By Proposition 9, $\vee(K^1, \dots, K^n) \in \mathcal{K}_P$. As this is true for all sequences $(K^1, \dots, K^n) \in \text{Seq}(\mathcal{K}_P)$, the distributed knowledge aggregator is partition-preserving.

Consider the **common knowledge** aggregator $\wedge$. For ease of notation, write $J = (K^1, \dots, K^n) \in \text{Seq}(\mathcal{K}_P)$, and $\text{Seq}(J)$ for the set of finite sequences of agents $j \in J$. As each K^j is partitional, let π^j be the partition which represents K^j according to Equation 5. That is, $K^jE = \{\omega \in \Omega \mid \pi^j(\omega) \subset E\}$. Let $\pi^{\wedge(J)}$ be the partition of Ω which is the finest common coarsening of all partitions π^j. That is $\pi^{\wedge(J)}(\omega) \supset \pi^j(\omega)$ for all $\omega \in \Omega$ and $j \in J$; and $\pi^{\wedge(J)}$ is the finest partition with this property. The finest common coarsening is given by

$$\pi^{\wedge(J)}(\omega) = \bigcup_{(j_1,\ldots,j_n)\in\mathrm{Seq}(J)} \pi^{j_1}\left(\cdots\left(\pi^{j_n}\left(\omega\right)\cdots\right)\right)$$

We will show that

$$\wedge^J E = \{\omega \in \Omega \mid \pi^{\wedge(J)}(\omega) \subset E\}$$

By Proposition 9, each K^j satisfies Monotonicity and Conjunction, so $K^j(E \cap F) = K^jE \cap K^jF$ for all events $E, F \in 2^\Omega$ and agents $j \in J$. Therefore,

$$\wedge^J E = \bigcap_{s=1}^{\infty} \left(\curlywedge^J\right)^{(s)} E = \bigcap_{(j_1,\ldots,j_n)\in\mathrm{Seq}(J)} K^{j_n}\cdots K^{j_1}E$$

So $\omega \in \wedge^J E$ if and only if $\omega \in K^{j_n}\cdots K^{j_1}E$ for all $(j_1, \ldots, j_n) \in \mathrm{Seq}(J)$.

Fix $E \in 2^\Omega$. By Equation 5, for all $\omega \in \Omega$,

$$\omega \in K^{j_1}E \iff \pi^{j_1}(\omega) \subset E$$

Suppose, for the purpose of induction, that for all sequences $(j_1, \ldots, j_{n-1})$ of length $n-1$, and all $\omega \in \Omega$,

$$\omega \in K^{j_{n-1}}\cdots K^{j_1}E \iff \pi^{j_1}\left(\cdots\left(\pi^{j_{n-1}}\left(\omega\right)\right)\right) \subset E$$

Then by Equation 5,

$$\begin{aligned}\omega \in K^{j_n}\cdots K^{j_1}E &\iff \pi^{j_n}(\omega) \subset K^{j_{n-1}}\cdots K^{j_1}E \\ &\iff \left[\tilde{\omega} \in \pi^{j_n}(\omega) \implies \tilde{\omega} \in K^{j_{n-1}}\cdots K^{j_1}E\right]\end{aligned}$$

Using the inductive hypothesis,

$$\begin{aligned}&\iff \left[\tilde{\omega} \in \pi^{j_n}(\omega^*) \implies \pi^{j_1}\left(\cdots\left(\pi^{j_{n-1}}\left(\tilde{\omega}\right)\right)\right) \subset E\right] \\ &\iff \left[\pi^{j_1}\left(\cdots\left(\pi^{j_{n-1}}\left(\tilde{\omega}\right)\right)\right) \subset E \text{ for all } \tilde{\omega} \in \pi^{j_n}(\omega)\right] \\ &\iff \bigcup_{\tilde{\omega}\in\pi^{j_n}(\omega)} \pi^{j_1}\left(\cdots\left(\pi^{j_{n-1}}\left(\tilde{\omega}\right)\right)\right) \subset E\end{aligned}$$

As the partition functions are set-valued functions,

$$\iff \pi^{j_1}\left(\cdots\left(\pi^{j_n}\left(\omega\right)\right)\right) \subset E$$

By induction over the length of the sequence of agents, for all sequences $(j_1, \ldots, j_n) \in \mathrm{Seq}(J)$, and all $\omega \in \Omega$,

$$\omega \in K^{j_n} \cdots K^{j_1} E \iff \pi^{j_1}\left(\cdots\left(\pi^{j_n}\left(\omega\right)\right)\right) \subset E$$

Aggregating over all sequences gives

$$\omega \in \bigcap_{(j_1,\ldots,j_n)\in \mathrm{Seq}(J)} K^{j_n} \cdots K^{j_1} E \iff \bigcup_{(j_1,\ldots,j_n)\in \mathrm{Seq}(J)} \pi^{j_n}\left(\cdots\left(\pi^{j_1}\left(\omega\right)\right)\right) \subset E$$

which is

$$\bigcap_{(j_1,\ldots,j_n)\in \mathrm{Seq}(J)} K^{j_n} \cdots K^{j_1} E = \left\{\omega \in \Omega \mid \bigcup_{(j_1,\ldots,j_n)\in \mathrm{Seq}(J)} \pi^{j_n}\left(\cdots\left(\pi^{j_1}\left(\omega\right)\right)\right) \subset E\right\}$$

Therefore

$$\wedge^J E = \{\omega \in \Omega \mid \pi^{\wedge(J)}(\omega) \subset E\}$$

as required. □

Proof of Proposition 12:

Let $(K^1, \ldots, K^n), (\tilde{K}^1, \ldots, \tilde{K}^n) \in \mathrm{Seq}(\mathcal{K}_1)$ such that $\tilde{K}^i$ is more informed than K^i for all $i = 1, \ldots, n$. Let $J = (K^1, \ldots, K^n)$ and $\tilde{J} = (\tilde{K}^1, \ldots, \tilde{K}^n)$. Fix an event $E \in 2^\Omega$. We want to show that the common knowledge aggregator $\wedge$ is increasing. That is,

$$\wedge^J E \subset \wedge^{\tilde{J}} E$$

From Lemma 2, the everybody knows aggregator $\curlywedge$ is increasing. Therefore $\curlywedge^J E \subset \curlywedge^{\tilde{J}} E$. Suppose, for the purposes of induction, that $\left(\curlywedge^J\right)^{(s)} E \subset \left(\curlywedge^{\tilde{J}}\right)^{(s)} E$ for some $s \in \mathbb{N}$. By Proposition 11, the aggregation $\curlywedge^J$ is monotonic. So

$$\left(\curlywedge^J\right)^{(s+1)} E = \curlywedge^J \left(\left(\curlywedge^J\right)^{(s)}\right) E \subset \curlywedge^J \left(\left(\curlywedge^{\tilde{J}}\right)^{(s)}\right) E$$

As $\curlywedge^{\tilde{J}}$ is more informative than $\curlywedge^{J}$,

$$\curlywedge^{J}\left(\left(\curlywedge^{\tilde{J}}\right)^{(s)}\right)E \subset \curlywedge^{\tilde{J}}\left(\left(\curlywedge^{\tilde{J}}\right)^{(s)}\right)E = \left(\curlywedge^{\tilde{J}}\right)^{(s+1)}E$$

By induction over $s \in \mathbb{N}$, for all $s \in \mathbb{N}$

$$\left(\curlywedge^{J}\right)^{(s)}E \subset \left(\curlywedge^{\tilde{J}}\right)^{(s)}E$$

Therefore

$$\wedge^{J}E = \bigcap_{s=1}^{\infty}\left(\curlywedge^{J}\right)^{(s)}E \subset \bigcap_{s=1}^{\infty}\left(\curlywedge^{\tilde{J}}\right)^{(s+1)}E = \wedge^{\tilde{J}}$$

As this hold for all $E \in 2^{\Omega}$, the common knowledge aggregator is increasing. □

Proof of Proposition 13:

Let $(K^1, \ldots, K^n) \in \text{Seq}(\mathcal{K})$ with $K^i \in \mathcal{K}_D$ for at least one $i \in 1, \ldots, n$. Let $\mathbb{A} : \text{Seq}(\mathcal{K}) \to \mathcal{K}$ be a negative aggregator, so that for all events $E \in 2^{\Omega}$, and all $i = 1, \ldots, n$:

$$A(K^1, \ldots, K^n)E \subset K^iE$$

Suppose, without loss of generality, that $K^1 \in \mathcal{K}_D$. Then

$$\left[A(K^1, \ldots, K^n)E\right] \cap \left[A(K^1, \ldots, K^n)\neg E\right] \subset K^1E \cap K^1\neg E = \emptyset$$

Therefore $\mathbb{A}(K^1, \ldots, K^n) \in \mathcal{K}_D$, as required. □

Proof of Proposition 14:

Let Ω be state space, and $\mathcal{K}$ the associated set of knowledge operators. Let $\mathbb{A} : \text{Seq}(\mathcal{K}) \to \mathcal{K}$ be a positive aggregator.

Choose two states $a, b \in \Omega$ with $a \neq b$. Define knowledge operators $K^1, K^2 \in \mathcal{K}$ by

$$K^1E = E, \qquad K^2E = \begin{cases} (E \setminus \{a\}) \cup \{b\} & ; \text{for } a \in E,\ b \notin E \\ (E \setminus \{b\}) \cup \{a\} & ; \text{for } a \notin E,\ b \notin E \\ E & ; \text{otherwise} \end{cases}$$

so that K^1 represents a fully informed agent, and K^2 represents an agent who is mostly fully informed, but mixes up state a and state b. By inspection, both operators K^1 and K^2 satisfy Axiom D. Moreover

$$\curlyvee(K^1, K^2)\{a\} = K^1\{a\} \cup K^2\{a\} = \{a\} \cup \{b\} = \{a, b\}$$
$$\curlyvee(K^1, K^2)\neg\{a\} = K^1\neg\{a\} \cup K^2\neg\{a\} = \neg\{a\} \cup \neg\{b\} = \Omega$$

Therefore $\curlyvee(K^1, K^2)\{a\} \cap \curlyvee(K^1, K^2)\neg\{a\} = \{a, b\} \neq \emptyset$.

By Proposition 4 the aggregator $\mathbb{A}$ is more informative than $\curlyvee$, so that for all events $E \in 2^\Omega$,

$$\mathbb{A}(K^1, K^2)E \supset \curlyvee(K^1, K^2)E$$

In particular,

$$\mathbb{A}(K^1, K^2)\{a\} \cap \mathbb{A}(K^1, K^2)\neg\{a\} \supset \{a, b\} \cap \Omega \supsetneq \emptyset$$

As $\mathbb{A}(K^1, K^2)\{a\} \cap A(K^1, K^2)\neg\{a\} \neq \emptyset$, the positive aggregator $\mathbb{A}$ does not preserve Axiom D. $\square$

References

[1] Anscombe, F. J., & Aumann, R.J., (1963) A Definition of Subjective Probability. *The Annals of Mathematical Statistics, 34*(1), 199-205.

[2] Aumann, R., & Brandenburger, A., (1995) Epistemic Conditions for Nash Equilibrium. *Econometrica, 63*(5), 1161-180.

[3] Bacharach, M., (1985) Some extensions to a claim of Aumann in an axiomatic model of knowledge. *Journal of Economic Theory, 37*(1), 167-190.

[4] Baltag A., Moss L.S., & Solecki S. (2016) The Logic of Public Announcements, Common Knowledge, and Private Suspicions. In: Arl-Costa H., Hendricks V., van Benthem J. (eds) *Readings in Formal Epistemology.* Springer Graduate Texts in Philosophy, vol 1. Springer, Cham

[5] Brandenburger, A., Dekel, E., & Geanakoplos, J. D., (1992) Correlated Equilibrium with Generalized Information Structures. *Games and Economic Behavior, 4*(2), 182-201.

[6] Berge, C., The Theory of Graphs, Dover (2001), republication of Methuen: John Wiley & Sons (1962 and 1966).

[7] Condorcet, M.J.A de, (1785), *Essai sur l'application de l'analyse à la probabilité des dcisions rendues à la pluralité des voix.* Paris: l'Imprimerie Royale [fascimile edition New York: Chelsea, 1972].

[8] Chellas, B. F., (1980) *Modal Logic: An Introduction.* Cambridge University Press.

[9] Dekel, E., Lipman, B. L., & Rustichini, A., (1998) Recent Developments in Modeling Unforeseen Contingencies. *European Economic Review, 42*, 523-542.

[10] Fagin, R. & Halpern, J., (1988) Belief, Awareness and Limited Reasoning. *Artificial Intelligence, 34*, 39-76.

[11] Feinberg, Y., (2000) Characterizing Common Priors in the Form of Posteriors, *Journal of Economic Theory, 91*, 127-179.

[12] Geanakoplos, J. D., (1989) Game Theory without Partitions, and Applications to Speculation and Consensus. *Cowles Foundation Discussion Paper No. 914.*

[13] Geanakoplos, J. D., (1992) Common Knowledge. *Handbook of Game Theory* (pp. 1437-1496). North Holland: Elsevier.

[14] Gorman, W. M., (1961) On a Class of Preference Fields. *Metroeconomica, 13*(2), 53-56.

[15] Harsanyi, J.C., Games with Incomplete Information Played by Bayesian Players, I, II, III, Manag. Sci. 14 (1967-1968), 159-182, 320-334, 486-502.

[16] Hart, S., Heifetz, A., & Samet, D., (1996) "Knowing Whether," "Knowing That," and The Cardinality of State Spaces. *Journal of Economic Theory, 70*(1), 249-256.

[17] Heifetz, A., Meier, M., & Shipper, B., (2006) Interactive Unawareness. *Journal of Economic Theory, 130*(1), 78-94.

[18] Lehrer, E., & Samet, D., (2011) Agreeing to Agree. *Theoretical Economics, 6*, 269-287.

[19] Lewis, D., (1969) *Convention: A philosophical study.* Harvard University Press.

[20] Machina, M.J., (1987) Choice Under Uncertainty: Problems Solved and Unsolved. *Economic Perspectives, 1*(1), 121-154.

[21] Machina, M.J., (2005) Choice Under Uncertainty. *Encyclopedia of Cognitive Science.* Nadel, N. (ed.) (vol. 1), 5050-514.

[22] Neiger, G., & Tuttle, M.R., (1993) Common Knowledge and Consistent Simultaneous Coordination. *Distributed Computing, 6*(3), 181-192.

[23] Pooya, M., Tahbaz-Salehi, A., & Jadbabie, A., (2018) A Theory of Non-Bayesian Social Learning. *Econometrica, 86*(2), 445-490.

[24] Osborne, M. & Rubinstein, A., (1994) *A Course in Game Theory.* The MIT Press.

[25] Rubinstein, A., (1998) *Modeling Bounded Rationality.* The MIT Press.

[26] Samuelson, L., (2004) Modeling Knowledge in Economic Analysis. *Journal of Economic Literature, 57*, 367-403.

[27] Samuelson, P. A., (1956) Social Indifference Curves. *The Quarterly Journal of Economics, 71*(1), 1-22.

[28] Savage, L., (1954) *The Foundation of Statistics.* Wiley Publications in Statistics.

[29] van Ditmarsch, H., Halpern, J.Y., van der Koek, J., & Kooi, B. (Ed.). (2015) *Handbook of Epistemic Logic*. Location: Milton Keynes, UK.

[30] von Neumann, J., & Morgenstern, O., (1944) *Theory of Games and Economic Behavior.* Princeton University Press.

[31] Wang, Y., Kuppusamy, L., & van Eijck, J., (2009) Verifying epistemic protocols under common knowledge. *Proceedings of the 12th Conference on Theoretical Aspects of Rationality and Knowledge.* 257-266.